A TREATISE

ON

FACTORIAL ANALYSIS,

ETC.

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ON

FACTORIAL ANALYSIS,

WITH THE

SUMMATION OF SERIES;

CONTAINING VARIOUS

NEW DEVELOPMENTS OF FUNCTIONS, &c.

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INTRODUCTION.

The greater part of this Work was written some years ago, and certain portions have been recently read before the Royal Society, for which the Author has received the thanks of that learned body. Series enter, more or less, into all our branches of analysis, as well as into many of the higher departments of physical science: such a subject, therefore, viewed in all its aspects and bearings, would require volumes for its adequate discussion. The ablest treatise we have, upon the summation of algebraic series, is, doubtless, Herschel's Examples of Finite Differences; after the profound and original views of this great mathematician, any attempt at further generalization would seem to be almost a fruitless task. The summation of series, by the method of definite integrals, first employed by Euler, has been extended by Poisson, W. S. B. Woolhouse, and other writers; upon this class of series I have not touched, being desirous of maintaining a sort of unity in this little work.

The publication of any thing in mathematics, which professes to be new, is almost sure to excite, in a certain class of men, the sneer of contempt, or that cold expression of heartless approbation, which, as it stops all inquiry, not unfrequently consigns to unmerited oblivion, ideas, which might have become the germ of great discoveries. Has mathematics attained its final development, that it should be deemed impossible for the humble labourer in the field of science to

pick up some gleanings of truth? However, it must be stated, that at a period when there is such a vast accumulation of mathematical truth, it would be presumption in any one to lay claim to strict originality; all, therefore, that the Author of the following Treatise can say is, that so far as he knows, the theorems are either new, or obtained by processes which are new. In particular, he would call attention to Theorems 35, 38, 40, 41, 42, 43, 45, 48, 49, 50, 51, 52, 56, 34, and the leading parts of the third and fourth sections. The equality expressed in 43, is an extension of Murphy's Theorem•; and 35, is a remarkable generalization of a series due to Poisson.* The symbol, [ ], of a factorial was, the author believes, first employed by Vandermonde, but by the addition of an accent it is, in the following Treatise, rendered more general in its signification, thereby enabling us to express the factorial, as well in an ascending, as in a descending form. Regarded in this light, this symbol is a more general form of a binomial with an exponent; for when the increment is taken zero, the factorial exponent becomes the ordinary one. The symbol, S, of summation, is employed, not only to indicate the sum of a series, but also to abbreviate the form, and point out the law of any complex expression. The connection between this symbol, and that of definite integration, is rendered apparent by Theorem 53.

The subject of series is one of great interest, and, independently of its utility as a branch of mathematics, is highly calculated to enlarge the understanding. It is easy, in many cases, to see how the sum of an infinite series of terms should be equal to a finite magnitude: thus, let a square be drawn to represent a unit, and a line dividing the square into two equal parts or halves; let one of these halves be divided into

* See De Morgan's great work on the Calculus, p. 244.

two equal parts, forming fourths, then one of these fourths into eighths, and so on without end, then it will be apparent that the whole square, or unit, is made up of the terms, one-half, one-fourth, one-eighth, and so on to infinity. Nor are our conceptions of infinity limited to mathematical quantities: some of our sublimest ideas involve the consideration of a series: thus, our conception of Deity is the summation of all that is vast, lovely, beneficent, and holy in the universe,—the function of which peopled immensity is the development,—the last incomprehensible link in the interminable chain of causation.

To those who are in the habit of considering differentiation as the universal solvent of all mathematical difficulties, not a few of the methods of investigation employed in this Work will appear tedious and operose. Taking for granted some of our great analytical principles and theorems, it becomes an easy task to establish, by a deductive process, formulæ of less generality. There is a charm in this deductive process, and a factitious, though an entire, confidence in the accuracy of our results; for we employ formulæ which have received the tribute of admiration from the genius of all nations, and which may have shed a light upon the works of nature, or given rise to some of the most magnificent discoveries in art. It is nevertheless true, that the too frequent use of these general forms, leads us to undervalue principles which would enable us to work out results, more simply, if not more elegantly, than by differentiation or integration. Such methods disguise the real difficulties of a problem; for while it is easy to understand the meaning and use of a general formula, every mind cannot, at once, appreciate the principles of reasoning upon which the demonstration of that formula is based. By the application of a rule, transformations are effected, which save the exercise of thought, and the labour

of development; but the rigid solution of a problem, or the complete demonstration of a theorem, calls into vigorous operation the slumbering energies of the higher intellect, and may thus exercise an important influence upon the whole of the student's career. The love of truth is a lamp to the soul, and sheds a holy light over the whole of our destiny; but this sentiment can only be duly cherished by habituating ourselves to exact methods of thought.

Pure algebra has been, for many years, an unproductive, because a comparatively neglected, field of inquiry; proud of the majesty of science, men of great genius have generally lived in a region of transcendentalism and abstraction, adding truth to truth, without deigning to bestow a thought upon humbler, though not less useful, walks of knowledge. So long as the student is confined to the relations of definite magnitudes, his path is easy and pleasant,—he can test the accuracy of his conclusions by an appeal to figures,—he can retrace the steps of his investigations, by new assumptions; but the moment he comes to consider the properties of quantities indefinitely extended or diminished, he passes into a field beyond the ordinary range of his associations, and then the inquiry appears to him wrapped up in all the mystery in which ideas of infinity are commonly involved. The difficulty, however, lies not in the mathematics, but in the metaphysics of the subject; for, viewed as a mathematical fact, nothing can be more simple than the doctrine of ultimate ratios, which arises out of the most simple laws of quantity; and if the student were led to deduce results, by the application of the principle, before any veil is thrown over the process by the hasty adoption of an arbitrary notation, the difficulty, which most students experience, in learning the differential calculus, would be to a great extent obviated. As great principles are rarely, if ever, at first discovered in their most

abstract form, so certain initiatory steps of instruction seem best calculated to convey to the mind of a student the knowledge of such abstract truths. In an age, therefore, when not only truth, but the processes by which truth has been attained, are subjects of inquiry, any attempt calculated to enforce the consideration of the more subtile principles of reasoning, ought to meet with a due amount of encouragement.

The fact, that number may be made up in different ways, is the great principle upon which all analytical truths are founded: our most splendid theorems are, therefore, little more than disguised truisms or identities, where the difference of form is owing either to the manner of grouping the elements, or to the circumstance of certain expressions being left in an undeveloped state; but, if every operation be performed, and the terms, on each side of the theorem, arranged according to the powers of any one of the elements, then we shall have, not only an equality of value, but an actual identity of form. As the element of light, modified by the surfaces upon which it falls, gives that endless variety of shade and colour to the objects of creation; so the simple principle referred to, gives to mathematical quantities an infinite variety of forms and modes of development. Mathematical propositions, therefore, are necessary truths; that is, they would remain true, even though there were no intelligencies in the universe to appreciate them: hence we cannot wonder at the certainty which attends such inquiries: and hence too the tendency of minds, exclusively mathematical, to trace everything in nature to the operation of abstract laws, whilst minds of even an inferior order, engaged in the contemplation of those laws and relations of matter which are discoverable by experiment and observation, are led to the firm conviction, that the universe is not like the development of a mathematical quantity, which could not have been otherwise than it

is, but that, on the contrary, it is a vast machine, evincing, in all its parts, selections and contingent adaptations, and therefore bearing unequivocal indications of the existence and superintendence of an Intelligent Cause, whose characteristics are, power, goodness, and truth, limitless in their measure, because they are boundless in the extent of their manifestation. Hence it may be inferred, that a philosophical education ought to bring into harmonious activity all the susceptibilities of our nature,—to cultivate a due appreciation of moral, as well as mathematical evidence,—to train the moral feelings, as well as the intellectual powers,—and to give the most enlarged conceptions of the universe, and of the relations of man to his present and future destinies.

Considered in its bearing on the progress of science, the importance of abstract notation can scarcely be over-rated. The symbols of chemistry represent with remarkable beauty the atomic composition of bodies, and enable us to determine the changes produced by chemical action, with all the precision which attends mathematical operations. Symbols, indeed, constitute a universal language, possessing a definiteness of which no other language can boast. A perfect notation, whilst it is concise, should be in keeping with acknowledged forms, and symbolize, as far as possible, the processes intended to be pointed out. Symbols sometimes have a twofold signification, representing either quantities, or operations on quantities, according as we view the symbol as a whole, or in its parts: for example, the S, used in this work, considered by itself, is the symbol of an operation to be performed upon the function to which it is prefixed, whilst the expression, taken as a whole, is a concise representation of a series of actual quantities. It is not a little curious, that although symbols of operation have no meaning, disconnected from the functions to which they are prefixed, yet operations may

be performed upon them, just as if they were actual magnitudes. Theorems 48 and 50, are examples of this kind. Doubtless, the future progress of analytical mathematics, will much depend upon the improvement of our notation. The history of the fluxional calculus in this country, ought to be a sufficient warning of the evils arising out of an undue attachment to an empirical and insufficient notation. It may be a question, whether the want of continuity in our branches of analysis be not, to a great extent, owing to an imperfection in our symbols. Lagrange's theory of derived functions may be ranked among the most decided improvements which have been made in the differential calculus; and if his assumption had been as logical in principle, as his views were simple and beautiful in exposition, little would have been desiderated on this subject. Yet we are led to hope, that some mind, with knowledge embracing the ample field of cognate science,—with genius keen and comprehensive,—with the subtilty to detect, and the faculty to correct what is erroneous, and supply what is defective,—will, one day, do for analysis what Euclid has done for geometry.

The future progress of science is a pleasing, and by no means an unprofitable subject of reflection. The history of the past gives us the most decided assurances as to the future. Our philosophy is the accumulation of centuries of toilsome inquiry. In our age, the steam toy of Hiero has assumed the form of a mighty engine, performing the labour of the world; the principle observed by Archimedes in the solution of an isolated problem, has become the most powerful instrument of mathematical research; the apparently useless properties of the cone have been made the index to the planetary motions; and the extensive and accurate astronomical observations made with the telescope, superseding the records of Ptolemy, have thrown new light upon the system of the

universe. New sciences have been born in our own times, rivalling, if not surpassing, in the brilliancy of their phenomena, anything that had been discovered in former ages. There can be no such thing as retrogression, for the science of one age becomes a sacred legacy bequeathed to that which succeeds. Nature is being interrogated in a thousand different ways, and her responses, through the aid of printing, are proclaimed in imperishable characters to the literati of the earth. Limited by no geographical boundary, the fellowship of knowledge is universal. The sudden influx of barbarism may vitiate the genius or mutilate the records of a particular people, but it is impossible that such a moral catastrophe can visit, at one time, the widely-scattered nations devoted to the advancement of truth. The human mind is formed for indefinite progression: the universe, the manifestation of the Eternal Mind, contains exhaustless treasures of harmony, upon which the intellect of man may exercise itself; and, as the mathematics of an age may be regarded as the exponent, as well as the interpreter, of the science of that age, so we cannot have any great advancement of the one without a corresponding extension of the other. It is consolatory to believe, that amid the strife and tumult and changes of the world,—that whilst despots, in certain parts of the globe, are seeking the means of rivetting more securely the manacles of slavery,—philosophy, in its most extensive sense, is making a silent and unobtrusive progress, which will one day spread the blessings of peace, order, and rational liberty over the earth.

Battersea, 1844.

FACTORIAL ANALYSIS,

WITH

THE SUMMATION OF ALGEBRAIC SERIES.

SECTION I.

NOTATION.

(1.) In order, as much as possible, to abbreviate expressions, and with the view also of facilitating the operations, the following notation has been adopted:—

For $n + 1$ factorials of the form $(e + rx)$ $(a + r(x + 1))$ .... $(a + r(x + n))$ is put $[a + rx]_r^{n+1}$; where the affix r indicates the increment or common difference of the factors: or, if u be put for $a + rx$, the factorial will be more concisely expressed by $[u]_r^{n+1}$. In like manner, $x^m (x + r)^m$ .... $(x + nr)^m$ is written $[(x)^m]_r^{n+1}$; or, generally, $f(x) . f(x+1)$ .... $f(x + nr)$ is expressed by $[fx]_r^{n+1}$. When the increment of the factors is 1, the accent at the bottom of the bracket is omitted; thus, $[1]^n$ is put for $1 . 2 . 3 .... n$; but, as this quantity is of frequent occurrence, for greater convenience it will also be expressed by $n\,!$.

A series of quantities connected by the sign of addition, and successively derived from the general, or xth term, is defined by the symbol of summation annexed to the expression for the xth term; thus, $S_v^n x^3$ expresses a series of terms taken between the limits $x = n$, and $x = v$. Again, as an

example of double summation, $S^1_x S^1_x x^n$ is used as the symbol of $1^n + (1^n + 2^n) + (1^n + 2^n + 3^n) + \ldots . (1^n + 2^n \ldots . x^n)$.

Until the student has made himself familiar with the notation, he may find a little difficulty in following some of the operations. In all such cases, however, he is recommended to write out the factorials, &c. according to the ordinary form.

SECTION II.

ELEMENTARY OPERATIONS.

(2.) *To change the Sign of the Increment.*

It is obvious that the same factorial may be formed, either by commencing with the highest or lowest term: in the first case the increment is positive, whilst in the second it is negative; hence,

$$[u]^n_r = [u + (n-1)\,r]^n_{-r}\ ;\ \text{and}\ [1]^n = [n]^n_{-1},\ \text{or}\ n\,!$$

(3.) *To change the Magnitude of the Increment.*

$$[u]^n = \frac{1}{r^n}\,[ru]^n_r = \frac{1}{r^n}\,[z]^n_r,\ \text{putting}\ z\ \text{for}\ ru.$$

This formula changes the increment from 1 to r; and, for performing the converse operation, we have,

$$[u]^n_r = r^n \left[\frac{u}{r}\right]^n \text{and when}\ u = r,\ \text{we have}\ [r]^n_r = [1]^n\, r^n\,;$$

generally, $[u]^r_n = \left[\frac{vu}{r}\right]^n_v \div \left(\frac{v}{r}\right)^n = \frac{r^n}{v^n}[z]^n_v$; which changes the increment from r to v.

Similarly, $v^n\ [u]^n_r = [vu]^n_{vr}$.

Hence we observe, that whatever propositions may hereafter be established in reference to factorials having unity as the increment, all such propositions may be readily extended to the general form of the factorial.

(4.) *To add, multiply, divide, and otherwise reduce Factorials.*

In the examples given in the following part of this Section, although the accent is not printed, the increment is supposed to be r, excepting where mention is made to the contrary.

EXAMPLES.

1. $a\,[u]^m + b\,[u]^m = (a+b)\,[u]^m$.
2. $[u]^m + [u]^{m+1} = (1+u+mr)\,[u]^m$.
3. $[u+r]^n - [u]^n = [u+r]^{n-1}\,(u+nr-u) = nr\,[u+r]^{n-1}$.
4. $a\,[u]^n \times b\,[u+nr]^m = ab\,[u]^{n+m}$; because the factors are consecutive.

Let $m=m-n$; then,

$$a\,[u]^n \times b\,[u+nr]^{m-n} = ab\,[u]^m.$$

5. $[u]^n \times [u+tr]^m = [u]^t\,[u+tr]^{n-t} \times [u+tr]^m = [u]^{m+t}\,[u+tr]^{n-t}$.

6. $[u]^n\,[u+(n+t)r]^m = \dfrac{[u]^{n+t+m}}{[u+nr]^t}$

7. By successive decomposition,—

$[u]^n = [u]^m\,[u+mr]^{n-m} = [u]^m\,[u+mr]^p\,[u+(m+p)\,r]^{n-m-p}$
$=$ &c.

8. $[u]^n_{-r}\,[u+r]^m_r = [u+mr]^{n+m}_{-r}$.

9. $[u]^m \div [u]^n = [u]^n\,[u+nr]^{m-n} \div [u]^n = [u+nr]^{m-n}$; if $n=m-n$, then

$$[u]^m \div [u]^{m-n} = [u+(m-n)\,r]^n$$

10. $\dfrac{[u+mr]^n}{[u]^n}+\dfrac{[u+nr]^m}{[u]^m}=\dfrac{2\,[u]^{m+n}}{[u]^n\,[u]^m}=\dfrac{2\,[u+nr]^m}{[u]^m}$

by reducing to the same denominator, adding, &c.

11. $[u-r]^{n+1} \div [u]^{n-2} = (u-r)\,[u]^n \div [u]^{n-2} = (u-r)$ $[u+(n-2)\,r]^2$

12. $[u+mr]^{n-m} \div [u]^n = [u+mr]^{n-m} \div [u]^m\,[u+mr]^{n-m}$ $=\dfrac{1}{[u]^m}$

(5.) *To find the Product of a Series of Factorials having the same Factorial Exponent.*

$$[u]^n\,[v]^n\,[p]^n\,\ldots\ldots = [u\,.\,v\,.\,p\,\ldots\ldots]^n;$$

where the increment applies to all the quantities within the bracket.

If $v=u+r$, $p=u+2r$, and so on; then,

$[u]^n\,[u+r]^n\ldots\ldots[u+(m-1)\,r]^n = [\,[u]^m]^n$ or $[\,[u]^n]^m$

When m is a common exponent, we have,

$$(\,[u]^n)^m = [u^m]^n$$

These results, relative to factorial exponents, are remarkably analogous to the corresponding properties of ordinary exponents.

(6.) *To decompose a Factorial into Two others, connected by the Sign of Subtraction.*

1. $[u]^n = [u]^n\left\{\dfrac{(u+nr)-(u-r)}{(n+1)\,r}\right\} = \dfrac{1}{(n+1)\,r}\,(\,[u]^{n+1} -$ $[u-r]^{n+1})$; and when the increment is minus,

$$[u]^n_{-r} = \frac{1}{(n+1)\,r}(\,[u+r]^{n+1}_{-r} - [u]^{n+1}_{-r}$$

2. In like manner it is shown that

$$\frac{1}{[u]^{n+1}} = \frac{1}{nr}\left(\frac{1}{[u]^n} - \frac{1}{[u+r]^n}\right); \text{ and}$$

$$\frac{1}{[u]_{-r}^{n+1}} = \frac{1}{nr}\left(\frac{1}{[u-r]_{-r}^n} - \frac{1}{[u]_{-r}^n}\right)$$

(7.) *To find certain particular Forms, and Limiting Values of Factorials.*

1. If $r = 0$, $[u]_0^n = u^n$; that is, in this case, the factorial figure becomes an exponent.

The same result will follow if u be taken infinite as compared with r.

2. The factorial will become nothing, for all values of u between 0 and $-(n-1)\,r$, for any of those values will introduce a factor which equals 0. Hence also all such values of u, render $\frac{1}{[u]^n}$ infinite.

3. Two factorials may have a finite ratio, although they may contain an infinite term; thus, in the ratio $[a + u]^n \div [b + u]^n$ let u be taken infinite, then we shall have, $u^n \div u^n = 1$, so that the ratio is that of equality.

4. By changing the sign of the exponent of p, we have $\frac{[u]^n}{[p]^n} = [u \,.\, p^{-1}]^n$.

(8.) *To change the Sign of the Factorial Exponent.*

For the remaining part of this section the increment is supposed to be unity.

1. In the expression, $[u]^n\,[u+n]^{m-n} = [u]^m$, (see 4, (4),) take $n = 0$, then

$[u]^0\,[u]^m = [u]^m$; whence by division, $[u]^0 = 1$.

2. Take $m = 0$, in the same expression; then by the last result the right-hand member becomes unity, therefore, by division,

$$[u+n]^{-n} = \frac{1}{[u]^n},$$

making $u = 0$, $[n]^{-n} = \frac{1}{[0]^n} = \infty$;

or, putting $u-n$ for u,

$$[u]^{-n} = \frac{1}{[u-n]^n};$$

let $n = 0$, then, $[u]^{-0} = \frac{1}{[u]^0} = 1.$

3. Similarly,

$$[u]_{-1}^{-n} = \frac{1}{[u+n]_{-1}^n} = \frac{1}{[u+1]^n},$$

taking $u = 0$, we have $[0]_{-1}^{-n} = \frac{1}{[1]^n}$,

putting $u-1$ for u,

$$[u-1]_{-1}^{-n} = \frac{1}{[u]^n}.$$

4. Comparing 2 with 3, we have

$$[u+n]^{-n} = [u-1]_{-1}^{-n},$$

or writing $u+1$ for u,

$$[u+n+1]^{-n} = [u]_{-1}^{-n}.$$

(9.) *To express the Sign* $\pm$ *by a Factorial.*

Take $u = 0$ in the factorial $[u-m]^n$, where m is equal to or greater than n; then $[-m]^n$, will be plus or minus according as n is even or odd, for all the factors will be minus; if $m = n$, then $[-n]^n = [-1]_{-1}^n = \pm 1.2.3 \ldots n.$

By putting $u = 0$, in 2 (8), we find the following form of expression for this quantity, viz:—

$$[0]^{-n} = \frac{1}{[-n]^n}, \text{ or } [-n]^n = \frac{1}{[0]^{-n}}$$

(10.) *To find the Interpretation of Expressions, when the Factorial Exponent is a Fraction, or an Infinite Quantity.*

$$[u]^n = [u]^{n+r} \div [u+n]^r = [u+r]^n [u]^r \div [u+n]^r,$$

putting p for u,

$$[p]^n = [p+r]^n [p]^r \div [p+n]^r$$

dividing the former by the latter,

$$\frac{[u]^n}{[p]^n} = \frac{[u+r]^n}{[p+r]^n} \times \frac{[u]^r [p+n]^r}{[p]^r [u+n]^r}.$$

Let r be taken infinite, then by (7) the limit of the first factor in the right-hand member is unity,

$$\therefore \frac{[u]^n}{[p]^n} = \frac{[u]^r [p+n]^r}{[p]^r [u+n]^r}.$$

Some curious expressions for numeral fractions may be derived from this formula.

1. Take $u=2$, $p=3$, and $n=2$, then,

$$\frac{1}{2} = \frac{[2]^r [5]^r}{[3]^r [4]^r} = \frac{2.5.3.6.4.7\ldots}{3.4.4.5.5.6\ldots},$$

which may easily be explained on other principles.

2. Take $u = 1$, $p = \frac{1}{2}$, and $n = \frac{1}{2}$, then

$$\frac{[1]^{\frac{1}{2}}}{[\frac{1}{2}]^{\frac{1}{2}}} = \frac{[1]^r [1]^r}{[\frac{1}{2}]^r [\frac{3}{2}]^r} = \frac{2.2.4.4.6.6\ldots}{1.3.3.5.5.7\ldots} = \frac{\pi}{2},$$

which is a remarkable factorial expression for the quadrant of a circle.

3. Let $u = \frac{7}{6}$, $p = \frac{1}{2}$, and $n = \frac{1}{3}$, then,

$$[\tfrac{7}{6}]^{\frac{1}{3}} \div [\tfrac{1}{2}]^{\frac{1}{3}} = \frac{7\,.\,5\,.\,13\,.\,11\,.\,19\,.\,17\,\ldots\ldots}{3\,.\,9\,.\;9\;.\,15\,.\,15\,.\,21\,\ldots\ldots}.$$

Here it will be observed, that as we advance in the series, the ratio of the factor in the numerator to the corresponding factor in the denominator, approaches to that of equality; hence, by actual calculation, the value of the factorial may be shown to be equal to $\frac{3}{2}$ as its limiting value.

The general relation of such factorials will be subsequently shown.

4. In the identity $[u]^n = [u]^m \div [u + n]^{m-n}$, take $u = \frac{1}{2}$, $n = \frac{1}{2}$, and $m = 1$, then,

$$[\tfrac{1}{2}]^{\frac{1}{2}} = \tfrac{1}{2} \div [1]^{\frac{1}{2}}; \;\therefore\; 2\,([\tfrac{1}{2}]^{\frac{1}{2}})^2 = [\tfrac{1}{2}]^{\frac{1}{2}} \div [1]^{\frac{1}{2}}.$$

Comparing with 2,

$$\frac{1}{2([\frac{1}{2}]^{\frac{1}{2}})^2} = \frac{\tau}{2}; \text{ and } \frac{1}{[\frac{1}{2}]^{\frac{1}{2}}} = \sqrt{\tau}.$$

5. The student will now have little difficulty in establishing the following identities:

$$[0]_{-1}^{\frac{1}{2}}\,[0]_{-1}^{-\frac{1}{2}} = [\tfrac{1}{2}]^{\frac{1}{2}} \div [1]^{\frac{1}{2}}; \;\; [\tfrac{1}{2}]^{\frac{1}{2}}_{-1} = \frac{1}{2} \div [\tfrac{1}{2}]^{\frac{1}{2}}$$

6. We proceed now to determine certain trigonometrical expressions for factorials.

Put $\frac{3}{2} - n$ for u, and $\frac{1}{2}$ for p, in (10), then

$$\frac{[\frac{3}{2}-n]^n}{[\frac{1}{2}]^n} \text{ or } \frac{[\frac{1}{2}]_{-1}{}^n}{[\frac{1}{2}]^n} = \frac{\frac{1}{2}\times -\frac{1}{2}\ldots\frac{2n-3}{2}}{\frac{1}{2}\times\frac{3}{2}\ldots\frac{2(n+1)-3}{2}} = \pm\frac{1}{2n-1}$$

where the sign is plus or minus according as n is odd or even.

Now $\sin (2n-1)\frac{\tau}{2} \div (2n-1) = \pm\frac{1}{2n-1}$, for all integral values of n.

Therefore the factorial above found will have an infinite number of values in common with this trigonometrical function; hence, therefore, by an extension of the signification of the factorial exponent, the development of the factorial as given in (10) may be put equal to this trigonometrical expression. Let therefore, for example, $n = \frac{1}{4}$, then $\frac{\sin -\frac{\tau}{4}}{-\frac{1}{2}} = \sqrt{2}$; and $\therefore \frac{[\frac{3}{2}-\frac{1}{4}]^{\frac{1}{4}}}{[\frac{1}{2}]^{\frac{1}{4}}} = \frac{3.5.7.\ 9\ldots.}{2.6.6.10\ldots.} = \sqrt{2}$; where it will be observed that $u = \frac{5}{4}$; $n = \frac{1}{4}$; and $p = \frac{1}{2}$.

Let $n = \frac{5}{6}$; then, $\frac{\sin 60^0}{\frac{2}{3}} = \frac{3}{4}\sqrt{3} = \frac{8.4.14.10.20.16\ldots}{3.9.9.15.15.21\ldots}$

By taking $n = \frac{3}{5}$; and observing that $\sin 18^0 = \frac{\sqrt{5}-1}{4}$, we shall have,

$$\frac{\sqrt{5}-1}{4} = \frac{11.9.21.19.31.29\ldots}{10.17.30.27.50.37\ldots}$$

In the same manner, Wallis' theorem, which is assumed in 4, may be readily established.

Similarly, also, the series of factors, 3, may be shown equal to $\frac{3}{2}$.

These trigonometrical values of factorials may be verified by actually calculating the factorial series as explained in 3; hence we may regard,

$$\sin (2n-1)\frac{\tau}{2} \div (2n-1) = \frac{[\frac{3}{2}-n]^r\,[\frac{1}{2}+n]^r}{[\frac{1}{2}]^r\,[\frac{3}{2}]^r} = \frac{[\frac{3}{2}-n]^n}{[\frac{1}{2}]^n}.$$

This formula, therefore, will give us expressions for the sines of various arcs. At the same time it is necessary to observe, that the series approximates too slowly to be of use in practical calculation. For example, in determining the sine of 10^0, it will be found, that we must take 20 factors, in order to have the result true to the second figure inclusive.

These relations may be further confirmed by an investigation of the properties of the symbol Γx.

7. Resuming the general expression (10), let us take $n = \frac{n}{m}$, $u = \frac{3}{2} - n$, and $p = \frac{1}{2}$; then

$$\frac{\left[\frac{1}{2}\right]^{\frac{n}{m}}_{-1}}{\left[\frac{1}{2}\right]^{\frac{n}{m}}} = \frac{\left[\frac{3}{2}-\frac{n}{m}\right]^r\left[\frac{1}{2}+\frac{n}{m}\right]^r}{\left[\frac{1}{2}\right]^r\left[\frac{3}{2}\right]^r}$$

If $m - n$ be put for n, the right-hand member will still express the same series.

$$\therefore \frac{\left[\frac{1}{2}\right]^{\frac{n}{m}}_{-1}}{\left[\frac{1}{2}\right]^{\frac{n}{m}}} = \frac{\left[\frac{1}{2}\right]^{\frac{m-n}{m}}_{-1}}{\left[\frac{1}{2}\right]^{\frac{m-n}{m}}}$$

These factorials possess this remarkable property in common with the trigonometrical function made use of in 6.

SECTION III.

(11.) Theorem.—Any function of x may be developed in the form $A_0 + A_1 [x]_r^1 + A_2 [x]_r^2 + \ldots$; where A_0, &c., are functions of y only.

It has been shown, on algebraic principles, that we may assume the proposed function $= fx + ax + bx^2 + \ldots$ Now the form of the development assumed in the theorem may be reduced to this latter, by expanding the factorials, and arranging the terms according to the ascending powers of x.

(12.) Theorem.—If we assume $u_x = A_0 + A_1 [x]_r^1 + \ldots A_n [x]_r^n$, where u_x is a function of x, not exceeding the nth degree, and determine the co-efficients so as to satisfy $n+1$ values of x, then the equation will hold true for any other values of x, that is, the expressions will be identical. Because the $n+1$ resulting equations, will serve to determine all the assumed co-efficients.

(13.) Theorem.—$u_x = \frac{1}{[1]^n r^n} \{a_0 [x+r]_r^n + a_1 [x]_r^1 [x+2r]_r^{n-1} + a_2 [x]_r^2 [x+3r]_r^{n-2} + \ldots a_n [x]_r^n\}$; where the value of the general co-efficient, $a_m = \frac{[-n]^m}{[1]^m} u_{-mr}$.

Assume

$$u_x = A_0 [x+r]_r^n + A_1 [x]_r^1 [x+2r]_r^{n-1} + \ldots A_n [x]_r^n.$$

Take $x = -mr$, then all the terms excepting the $(m+1)$th will vanish, and,

$$\therefore A_m = \frac{u_{-mr}}{[-mr]_r^m\,[r]_r^{n-m}} = \frac{[-n]^m}{[1]^m}\,\frac{u_{-mr}}{[r]_r^n};$$

by multiplying numerator and denominator by $[nr]_{-r}^m$, dividing by r^m, changing the sign of the minus increment, and then proceeding as in Ex. 4, (4). Whence the theorem becomes manifest.

(14.) THEOREM.—
$$\frac{u_x}{[x]_r^{n+1}} = \frac{1}{[1]^n r^n}\left\{u_0\frac{1}{x} - nu_{-r}\frac{1}{x+r} + \frac{n(n-1)}{1.2}u_{-2r}\frac{1}{x+2r} - \ldots \pm u_{-nr}\frac{1}{x+nr}\right\}.$$

This identity is established from the last theorem, by dividing by $[x]_r^{n+1}$, and writing the ordinary form of the binomial co-efficients. See (9).

Similarly and more generally,

$$\frac{fx}{[a+rx]_r^{n+1}} = \frac{1}{[1]^n r^n}\left\{K_0\frac{1}{a+rx} - nK_1\frac{1}{a+r(x+1)} + \frac{n(n-1)}{1.2}K_2\frac{1}{a+r(x+2)} - \ldots\right\} \text{ where } K_m = f\left(-\frac{a+mr}{r}\right).$$

Ex. 1. $$\frac{x^2+2}{[x]_3^3} = \frac{1}{[1]^2 3^2}\left\{2\frac{1}{x} - 2.11\frac{1}{x+3} + 38\frac{1}{x+6}\right\}$$

Ex. 2. $$\frac{4x^2+1}{[1+2x]_2^3} = \frac{1}{[1]^2 2^2}\left\{2\frac{1}{1+2x} - 2.10\frac{1}{3+2x} + 26\frac{1}{5+2x}\right\}$$

COR. Let $u_x = 1$, then,

$$\frac{1}{[x]_r^{n+1}} = \frac{1}{[1]^n r^n}\left\{\frac{1}{x} - n\frac{1}{x+r} + \frac{n(n-1)}{1.2}\frac{1}{x+2r} - \ldots\right\}$$

A more general form of decomposition is given in (16).

(15.) THEOREM.—$[y+x]_r^n=[y]_r^n+n[y+r]_r^{n-1}[x]_{-r}^1+\frac{n(n-1)}{1.2}[y+2r]_r^{n-r}[x]_{-r}^2+\ldots$

Assume $[y+x]_r^n=A_0+A_1[x]_{-r}^1+\ldots A_n[x]_{-r}^n$.

Take $x=0$, then $A_0=[y]_r^n$; take now $x=x+r$, and subtract the original equation from the resulting one; then we have,

$[y+x+r]_r^n-[y+x]_r^n=A_1r+A_2\{[x+r]_{-r}^2-[x]_{-r}^2\}+\ldots$
taking the difference of these factorials by 3, (4).

$[y+x+r]_r^{n-1}\times nr=A_1r+A_2[x]_{-r}^1\times 2r+A_3[x]_{-r}^2\times 3r+\ldots$; let now $x=0$, then $A_1=n[y+r]_r^{n-1}$.

The expression for A_2 is obtained from the last identity, precisely in the same way as A_1 has been derived from the original assumed identity; and so on to the other co-efficients.

1. For x put $-x$; then
$[y-x]_r^n=[y]_r^n-n[y+r]_r^{n-1}[x]_r^1+\frac{n(n-1)}{1.2}[y+2r]_r^{n-2}[x]_r^2-\ldots$

2. Let $r=0$ in the last and preceding results; then, the factorial indices becoming powers, we shall have,

$$(x\pm y)^n=y^n\pm ny^{n-1}x+\frac{n(n-1)}{1.2}y^{n-2}x^2\pm\ldots$$

which establishes the binomial theorem when n is an integer.

3. Writing the factorials so as to have all the increments minus (2), and then putting $y-(n-1)r$ for y, we shall have,

$$[y+x]_{-r}^n=[y]_{-r}^n+n[y]_{-r}^{n-1}[x]_{-r}^1+\ldots$$

Multiplying each side of this identity by $[x+nr]_{-r}^n$,
$[y+x]_{-r}^n[x+nr]_{-r}^n=[y]_{-r}^n[x+nr]_{-r}^n+n[y]_{-r}^{n-1}[x+nr]_{-r}^{n+1}+\ldots$

(16.) THEOREM.—$\frac{1}{[x]_r^{n+m}} = \frac{1}{[m]^{n}r^{m}}\left\{\frac{1}{[x]_r^m} - n\frac{1}{[x+r]_r^m} + \frac{n(n-1)}{1.2}\cdot\frac{1}{[x+2r]_r^m} - \ldots\right\}$

In No. 1, (15), put $x+mr$ for y; then divide by $[m]^{n}r^{m} \times [x]_r^{n+m}$; and the identity will be established.

This remarkable expression shows how a factorial fraction may be resolved into a series of fractions having a new factorial exponent.

EXAMPLES.

1. When $m=1$, this expression becomes the same as Cor. 1, (14).

2. Let $m=3$, then,

$$\frac{1}{[x]_r^{n+3}} = \frac{1}{[3]^{n}r^{n}}\left\{\frac{1}{[x]_r^3} - n\frac{1}{[x+r]_r^3} + \ldots \pm \frac{1}{[x+nr]_r^3}\right\}$$

(17.) THEOREM.

$$\frac{[y+x]_r^n}{[y]_r^n} = 1 + n\frac{[x]_{-r}^1}{[y]_r^1} + \frac{n(n-1)}{1.2}\,\frac{[x]_{-r}^2}{[y]_r^2} + \ldots$$

This result is established by dividing each side of Theorem (15), by $[y]_r^n$, and reducing the resulting expression by Ex. 12, (4).

By putting $x-y$ for x, we obtain an expression for the division of one factorial, by another of the same form.

1. Let $n=x$, and $r=1$, then,

$$\frac{[y+x]^x}{[y]^x} = 1 + \frac{x^2}{[y]^1} + \frac{x^2(x-1)^2}{1.2}\,\frac{1}{[y]^2} + \ldots$$

If $y=2$,

$$\frac{[2+x]^x}{[2]^x} = 1 + \frac{x^2}{1^2}\cdot\frac{1}{2} + \frac{x^2(x-1)^2}{1^2.2^2}\cdot\frac{1}{3} + \ldots$$

If $y=1$,

$$\frac{[1+x]^x}{[1]^x}=1^2+\frac{x^2}{1^2}\cdot+\frac{x^2(x-1)^2}{1^2.2^2}\quad\ldots$$

2. Let $n=-n$, then by 2 (8).

$$\frac{[y+x]^{-n}}{[y]^{-n}}=\frac{[y-n]^n}{[y+x-n]^n}=1-n\frac{[x]_{-1}{}^1}{[y]^1}+\frac{n(n+1)}{1.2}\frac{[x]_{-1}{}^2}{[y]^2}-\ldots$$

In 1, write $-x$ for x, and afterwards $y+x$ for y, then

$$\frac{[y]^x}{[y-x]^x}=1+x^2\frac{1}{[y+x]^1}+\frac{x^2(x+1)^2}{1.2}\frac{1}{[y+x]^2}+\ldots$$

Make $x=1$; then,

$$\frac{y}{y-1}=1+\frac{1}{y+1}+\frac{1.2}{(y+1)(y+2)}+\frac{1.2.3}{(y+1)(y+2)(y+3}+\ldots$$

Make $x=2$;

$$\frac{[y]^2}{[y-2]^2}=1+2^2\frac{1}{[y+2]^1}+\frac{2^2.3^2}{1.2}\cdot\frac{1}{[y+2]^2}+\ \ldots$$

In this result take $y=3$,

$$6=1+\frac{2^2}{1}\cdot\frac{1}{5}+\frac{2^2.3^3}{1.2}\cdot\frac{1}{5.6}+\ldots$$

These general expressions, though symbollically true, fail in some cases when numbers are substituted.

3. Writing in (17), $\frac{1}{2}$ for y, $1-x$ for x, and afterwards putting x for n,

$$\left[\frac{3}{2}-x\right]^x\div\left[\frac{1}{2}\right]^x=\frac{\sin(2x-1)\frac{\pi}{2}}{2x-1}=1+2[x][1-x]+\frac{2^2}{2.3}$$

$[x]_{-1}{}^2[1-x]_{-1}{}^2+\ldots$

Take $x = \frac{1}{2}$, then 2, (10).

$$\frac{\pi}{2} = 1 + \frac{1}{2} + \frac{1^2}{2.3} \cdot \frac{1}{2^2} + \frac{1^2.3^2}{2.3.3.5} \cdot \frac{1}{2^3} + \ldots$$

4. Proceeding with 1, (15), in the same manner, we have,

$$\frac{[y-x]_r^n}{[y]_r^n} = 1 - n\frac{[x]_r^1}{[y]_r^1} + \frac{n(n-1)}{1.2} \frac{[x]_r^2}{[y]_r^2} - \ldots$$

Neat expressions will be found by, 1° making $y=1$, $r=1$, $x=x+1$, and $n=x$, 2° making $y=1$, $r=1$, and $x=\frac{1}{2}$.

(18.) THEOREM.—$[y+nr]_r^{p-n} = \frac{1}{[-p]^n r^n}\Big\{ [y]_r^p - n [y+r]_r^p + \frac{n(n-1)}{1.2} [y+2r]_r^p - \ldots \Big\}$

This identity is obtained from 1, (15), by multiplying by $[y+nr]^{p-n}$, reducing the factorials, then putting $y+pr$ for x, and finally multiplying as in Ex. 4, (4).

This theorem may also be established, after the manner of proof employed in (15); or by the method of successive decomposition exhibited in (6).

1. Let $p=n$; then

$$[y]_r^n - n[y+r]_r^n + \frac{n(n-1)}{1.2} [y+2r]_r^n - \ldots = [-n]^n r^n.$$

This theorem will afterwards be given in a more general form.

(19.) THEOREM—$fx = A_0 + A_1 [y+rx]_r^1 + \ldots A_n[y+rx]_r^n + \ldots$; where $A_m = \frac{1}{[-m]^m r^m} \Big\{ f\Big(-\frac{y+mr}{r}\Big) - mf\Big(-\frac{y+\overline{m-1}r}{r}\Big) + \frac{m(m-1)}{1.2} f\Big(-\frac{y+m-2)r}{r}\Big) - \ldots \pm f\Big(-\frac{y}{r}\Big)\Big\}$

In the assumed expression take $x = -\frac{y+mr}{r}$, then all the terms after the mth will vanish, and we shall have,

$$A_m = \frac{1}{[-m]^{m} r^{m}} \left\{ f - \left(\frac{y+mr}{r}\right) - A_0 - A_1 [-m]^1 r - \ldots A_{m-1} [-m]^{m-1} r^{m-1} \right\}$$

In this expression take m successively $= 0, 1, 2, 3, \ldots$ then by elimination, &c.,

$A_0 = f\left(-\frac{y}{r}\right)$; and so on generally to the other co-efficients.

This theorem gives us the development of a function in the form of a factorial series.

1. If the increment r be taken minus, then the sign of r in the expression for A_m must be taken minus.

2. Make $y = 0$, and $rx = x$, and put u_x for fx; then, inverting the expression for the co-efficient, $u_x = A_0 + A_1 [x]_r^1 + \ldots A_n [x]_r^n + \ldots$; where $A_m = \frac{1}{[1]^m r^m}$ $\left\{ u_0 - mu_{-r} + \frac{m(m-1)}{1.2} u_{-2r} - \ldots \right\}$

The student is recommended to establish this result, after the manner of the general theorem. (See the observations in 1.)

3. Those who are acquainted with the subject of differences will not fail to observe, that when $r = -1$, the co-efficient $A_m = \Delta^m u_0 \div [1]^m$; and that in this case, 2, becomes the common theorem of differences.

4. Let $u_x = [x+r]_r^n$, then $u_0 = [1]^n r^n$; $u_{-r} = 0$, &c., then, $[x+r]_r^n = [x]_r^n + n [x]_r^{n-1} r + n(n-1) [x]_r^{n-2} r^2 + \ldots$

The same result may be obtained by putting r for x in Theorem (15).

5. Let $fx = x^n$, and $r = 1$,

$$x^n = (-y)^n - \{(\overline{-y+1})^n - (-y)^n\}\,[y+x]^1 + \ldots \text{ to } n \text{ terms.}$$

6. Put $y = 0$ in the last identity, and then,

$$x^n = -(-1)^n x + \frac{(-2)^n - 2(-1)^n}{1.2}\,[x]^2 - \ldots \text{ to } n \text{ terms.}$$

When $x = 1$, this identity becomes

$$1 = -(-1)^n + \{(-2)^n - 2\,(-1)^n\} - \ldots \text{ to } n \text{ terms.}$$

7. Let $x = 1$, $r = 1$, and $y = 0$ in the general theorem; then,

$$f(1) = A_n\,[1]^n + A_{n-1}\,[1]^{n-1} + \ldots A_0.$$

8. Let $u_x = a^x$, and $r = -1$, then $u_0 = 1$, $u_1 = a$, $u_2 = a^2$, &c., and,

$$a^x = A_0 + A_1\,[x]_{-1}{}^1 + A_2\,[x]_{-1}{}^2 + \ldots;$$

where $A_m = \dfrac{1}{[1]^m}\left\{a^m - ma^{m-1} + \dfrac{m(m-1)}{1.2}\,a^{m-2} - \ldots\right\} =$ $\dfrac{(a-1)^m}{[1]^m}$, because m is a whole number, and this development, in such cases, has been established in 2, (15).

Substitute these new values for the co-efficients, and then we shall have,

$$a^x = 1 + (a-1)x + \frac{(a-1)^2}{1.2}\,[x]_{-1}{}^2 + \ldots$$

Put now $a - 1 = z$, and $\therefore\ a = z + 1$,

$$\therefore\ (1+z)^x = 1 + x\,.\,z + \frac{x\,(x-1)}{1.2}\,z^2 + \ldots$$

which establishes the binomial theorem for all values of the exponent, for it is evident, from the manner in which Theorem

(19) has been derived, that x is not restricted in its signification.

(20.) Theorem.—$f(u) - nf(u+1) + \frac{n(n-1)}{1.2} f(u+2)$ $- \ldots = [-n]^n a_n$, where $f(u)$ is a function of the nth degree, and a_n is the co-efficient of u^n.

Because the development of $A_n\ [y+rx]_r^n$ in (19) will give us $A_n r^n$ for the co-efficient of x^n, therefore, comparing the co-efficients of the like powers, $A_n r^n = a_n$.

Let the value of A_n be now substituted in this equation, and let $r=-1$, and $y-n=u$; then, by an obvious reduction, the proposed theorem will be established.

1. Take $a_n=0$, that is, let $f(u)$ be a function of the $(n-1)$th degree, then,

$$f(u) - nf(u+1) + \frac{n(n-1)}{1.2} f(u+2) - \ldots = 0.$$

If $fu = a^m$, then

$$a^m - n(a+1)^m + \frac{n(n-1)}{1.2}(a+2)^m - \ldots = 0;$$

where m may not exceed $n-1$.

(21.) Theorem.—

$$\frac{u_x}{[v_x]_r^{n+1}} = \frac{A_0}{[v_x]_r^{n+1}} + \frac{A_1}{[v_{x+1}]_r^n} + \ldots \frac{A_{n-1}}{[v_{x+n-1}]_r^2};$$

where u_x is an expression of the $(n-1)$th degree, $v_x = y+rx$, and the co-efficients in the numerator are the same as in (19).

Divide each side of the general theorem (19) by $[v_x]_r^{n+1}$, and the above identity will be verified.

The student will observe, that as $v_x = y+rx$; so, therefore, $v_{x+n} = y + r(x+n)$.

(22.) Theorem.—$\frac{u_x}{[v_x]_r^{n+1}} = w_x - w_{x+1}$; where

$$w_x = \frac{A_0}{nr}\,\frac{1}{[v_x]_r^{\,n}} + \frac{A_1}{(n-1)^r}\,\frac{1}{[v_{x+1}]_r^{\,n-1}} + \ldots \frac{A_{n-1}}{r}\,\frac{1}{v_{x+n-1}}$$

This theorem is obtained from the preceding one, by decomposing each term by 2, (6).

(23.) THEOREM.—$u_x = K_x - K_{x-1}$; where

$$K_x = A_0[x]^1 + \frac{A_1}{2}[x]^2 + \ldots \frac{A_n}{n+1}[x]^{n+1}.$$

This result is derived from 2, (19), by decomposing each term of the left-hand member by 1, (6). Connecting the results of the two last theorems, it appears that any integral function may be resolved into the difference of two other functions, where the one is derived from the other by the substitution of $x \pm 1$ for x, and that the same holds true in reference to a rational fraction having factorials in the denominator.

(24.) THEOREM. If u_x be the general or xth term of a series, then

$$S_x^m u_x = A_0\{[x]^1 - [m-1]^1\} + \frac{A_1}{2}\{[x]^2 - [m-1]^2\} + \ldots$$
$$\frac{A_n}{n+1}\{[x]^{n+1} - [m-1]^{n+1}\}.$$

Because $S_x^m u_x = u_m + u_{m+1} + \ldots u_x$.

But by (23) each term may be decomposed by the formula $u_x = K_x - K_{x-1}$.

$\therefore S_x^m u_x = (K_m - K_{m-1}) + (K_{m+1} - K_m) + \ldots (K_x - K_{x-1}) = K_x - K_{m-1}$.

By substituting the value of K_x given in (23), the identity announced in the theorem is obtained.

1. When $m=1$, $S_x^1 u_x = A_0\,[x]^1 + \frac{A_1}{2}[x]^2 + \ldots$

(25.) Similarly,

$$S_x^m \frac{u_x}{[v_x]^{n+1}} = w'_m - w_{x+1},$$

where the symbols have the same signification as in (22,.

(26.) The results of the preceding theorem may also be derived by the following method.

As the required sum must be some function of x, let

$$S_x^1 u_x = a_0 + a_1 [x]^1 + \ldots a_n[x]^n + \ldots$$

$$\therefore S_{x+1}^1 u_x = a_0 + a_1 [x+1]^1 + \ldots a_n [x+1]^n + \ldots$$

But the latter is the expression for the series carried a term further than the former; therefore, by subtraction, (see 3, (4),)—

$$u_{x+1} = a_1 + 2a_2[x+1] + \ldots (n+1)\, a_{n+1}\, [x+1]^n$$

whence the co-efficients a_1, a_2, ... may be readily found by the method employed in the demonstration of Theorem (19), and a_0 will be determined by a consideration of the point at which the series begins.

(27.) It has been shown (16) how $\frac{1}{[x]_r^{n+1}}$ may be resolved into fractions having any number of factors in the denominator, and that the co-efficients of the decomposed terms are the co-efficients of the binomial development. Further, in 1, (20), it has been demonstrated that these binomial co-efficients are still equal to zero, after they have been respectively multiplied by expressions, not exceeding the $(n-1)$th degree, which are derived from each other by the substitution of $x \pm 1$ for x. These results are the foundation of the following method for the summation of certain fractional series.

As occasion may require, we shall in future put n_0, n_1, n_2, ..., for the binomial co-efficients when n is the exponent; then, by decomposing the successive terms of the series by (16) we have,

$$\left.\begin{array}{l}
\dfrac{u_x}{[v_x]_r^{n+1}} = n_0\dfrac{u_x}{v_x} - n_1\dfrac{u_x}{v_{x+1}} + n_2\dfrac{u_x}{v_{x+2}} - \ldots \mp n_1\dfrac{u_x}{v_{x+n-1}} \pm n_0\dfrac{u_x}{v_{x+n}} \\
\dfrac{u_{x+1}}{[v_{x+1}]_r^{n+1}} = \qquad n_0\dfrac{u_{x+1}}{v_{x+1}} - n_1\dfrac{u_{x+1}}{v_{x+2}} + \ldots \pm n_2\dfrac{u_{x+1}}{v_{x+n-1}} \mp n_1\dfrac{u_{x+1}}{v_{x+n}} \pm \ldots \\
\dfrac{u_{x+2}}{[v_{x+2}]_r^{n+1}} = \qquad\qquad n_0\dfrac{u_{x+2}}{v_{x+2}} - \ldots \mp n_3\dfrac{u_{x+2}}{v_{x+n-1}} \pm n_2\dfrac{u_{x+2}}{v_{x+n}} \mp \ldots \\
\quad\vdots \\
\dfrac{u_{x+n-1}}{[v_{x+n-1}]_r^{n+1}} = \qquad\qquad\qquad n_0\dfrac{u_{x+n-1}}{v_{x+n-1}} - n_1\dfrac{u_{x+n-1}}{v_{x+n}} + \ldots \\
\dfrac{u_{x+n}}{[v_{x+n}]_r^{n+1}} = \qquad\qquad\qquad\qquad n_0\dfrac{u_{x+n}}{v_{x+n}} - \ldots \\
\text{\&c.} \qquad\qquad\qquad\qquad\qquad\qquad + \text{\&c.}
\end{array}\right\} \times \frac{1}{[1]^{n_{\gamma}n}}$$

Here by 1, (20), the $(n+1)$th column, as well as all those which succeed, becomes zero, and hence, the sum of the series is equal to the terms which precede the $(n+1)$th column, that is,

$$S_{\infty}^{x}\frac{u_x}{[v_x]_r^{n+1}} = \frac{1}{[1]^{n_{\gamma}n}}\left\{n_0\frac{u_x}{v_x} + \frac{1}{v_{x+1}}(n_0u_{x+1} - n_1u_x) + \ldots + \frac{1}{v_{x+n-1}}(n_0u_{x+n-1} - n_1u_{x+n-2} + \ldots n_1u_x)\right\}.$$

(28.) THEOREM.—

$$[m]_{-1}^{n} \pm a_1 [m+1]_{-1}^{n} x + a_2 [m+2]_{-1}^{n} x^2 \pm \ldots =$$

$$[-n]^{n} \{x^n \pm a_1 b_1 x^{n-1} (1 \pm x) + a_2 b_2 x^{n-2} (1 \pm x)^2 \pm \ldots\};$$

where a_1, a_2, ... are the binomial co-efficients arising from the nth power, and b_1, b_2, ... those arising from the mth power.

By expansion and multiplication,

$$x^m (1 \pm x)^n = x^m \pm a_1 x^{m-1} + a_2 x^{m+2} \pm \ldots$$

Let x become $x+h$; then we shall determine an equation (α), where, if the expressions on the left-hand member be expanded, the co-efficient of h^n will be found to be the series within the vinculum, on the right-hand member of the theorem, multiplied by certain powers of x; but by expanding the right-hand member (see *Hymer's Theory of Equations*, p. 31). the co-efficient of h^n will be found to be the first part of the theorem, multiplied by certain powers of x, divided by $[-n]^n$; hence, by equating, the identity, stated in the theorem, is established.

1. When $x=1$, then when the bottom sign is taken,

$$[m]^n_{-1} - a_1 [m+1]_{-1}^{n} + \ldots = [-n]^n$$

and when the top sign is taken,

$$[m]_{-1}^{n} + a_1 [m+1]_{-1}^{n} + \ldots = [-n]^n \{1 + 2a_1 b_1 + 2^2 a_2 b_2 + \ldots\}$$

2. Dividing the second identity by the first,

$$\frac{[m]_{-1}^{n} + a_1 [m+1]_{-1}^{n} + \ldots}{[m]_{-1}^{n} - a_1 [m+1]_{-1}^{n} + \ldots} = 1 + 2a_1 b_1 + 2^2 a_2 b_2 + 2^3 a_3 b_3 + \ldots$$

3. Hence, also, we have,

$$[n]^2 - [n-1]^4 \frac{1}{1^2.2^2} + [n-2]^6 \frac{1}{1^2.2^2.3^2} - \ldots = 0 \text{ or } 2,$$

$[2n]_{-1}^{2n} - a_1\,[2n+1]_{-1}^{2n} + a_2\,[2n+2]_{-1}^{2n} - \ldots =$ $([-2n]_{-1}^{n})^2$.

A few examples are here given on the summation of series.

EXAMPLES.

1. $1^n+2^n+\ldots x^n = -\frac{(-1)^n}{2}\,[x]^2 + \frac{-2(-1)^n+(-2)^n}{2.3}$ $[x]^3+\ldots+\left(-n_1(-1)^n+n_2(-2)^n-n_3(-3)+\ldots\right)\frac{[x]^{n+1}}{[1]^{n+1}}$ by 1, (24), and 2, (19).

2. If $n=2$, the sum becomes, after an easy reduction, $\frac{x(x+1)(2x+1)}{6}$.

The following method of summing this series is highly eligible for elementary instruction.

Let $S_2 = 1^2+2^2+\ldots n^2$; and $S_1 = 1+2+\ldots n$; then in the expansion,

$$(x+1)^3 = x^3+3x^2+3x+1$$

take x successively $=1, 2, 3, \ldots n$, and add the resulting equations, and we shall have,

$$2^3 = 1^3+3\,.\,1^2+3\,.\,1+1$$
$$3^3 = 2^3+3\,.\,2^2+3\,.\,2+1$$
$$\vdots \qquad \vdots$$
$$(n+1)^3 = n^3+3\,.\,n^2+3\,.\,n+1$$
$$\overline{(n+1)^3 = 1^3+3\ S_2+3\ S_1+n}$$

Then, by solving this equation, and observing that

$$S_1 = \frac{n(n+1)}{2},\text{ we have } S_2 = \frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{6}$$

By expanding $(x+1)^4$, the sum of the fourth powers, may in like manner be determined in terms of S_2, and S_1; and so on generally.

3. To find $S^1_\infty \dfrac{x}{[x+2]_2{}^3}$.

Here, by (14),

$$\begin{array}{lllll}
\dfrac{1}{1.3.5} & = 1 - 2.\dfrac{1}{3} + \dfrac{1}{5} & & & \\
\dfrac{2}{3.5.7} & = & \dfrac{2}{3} - 2.\dfrac{2}{5} + \cdot\cdot & \left.\right\} \dfrac{1}{1.2.2^2} & \\
\dfrac{3}{5.7.9} & = & \quad\;\; \dfrac{3}{5} - \cdot\cdot & & \\
\cdot & & \quad\;\; \cdot & & \\
\cdot & & \quad\;\; \cdot & & \\
\cdot & & \quad\;\; \cdot & &
\end{array}$$

where the 3rd, 4th, &c., columns $= 0$, and the sum of the series, therefore, becomes $= \dfrac{1}{8}$.

4. To find $S^1_\infty \dfrac{2x+3}{[x]^4}$.

Decomposing by (14),

$$\begin{array}{llll}
\text{1st term} = \dfrac{5}{1.2} - 2.\dfrac{5}{2.3} + \dfrac{5}{3.4} & & & \\
\text{2nd term} = & \dfrac{7}{2.3} - 2.\dfrac{7}{3.4} + \cdot\cdot & \left.\right\} \dfrac{1}{2.3.1^2} & \\
\text{3rd term} = & \quad\;\; \dfrac{9}{3.4} - \cdot\cdot & & \\
\qquad\quad \cdot & \quad\;\; \cdot & & \\
\qquad\quad \cdot & \quad\;\; \cdot & & \\
\qquad\quad \cdot & \quad\;\; \cdot & &
\end{array}$$

Proceeding as before, the sum required will be found to be $\frac{1}{3}$.

In this case it is not necessary to decompose the terms into their simple fractions.

If, in such series, the factors in the denominator are not consecutive, the deficiency may be supplied, by multiplying the numerator and denominator by the factors that are wanting.

SECTION IV.

THE following section contains the investigation of different methods, for finding the sum of a certain class of recurring series.

(30.) THEOREM.—If u_x be a function of the $(n-1)$th degree, and a any proper fraction, then

$$\{u_{x+1}\, a^{x+1} + u_{x+2}\, a^{x+2} + u_{x+3}\, a^{x+3} + \text{ad. inf.}\} \times (1-a)^n$$

will produce a finite quantity.

Let the binomial be developed, and let the resulting product be arranged according to the powers of a, then the co-efficient of the general power a^{x+m}, will be,

$$u_{x+m} - nu_{x+m-1} + \frac{n\,(n-1)}{1.2}\, u_{x+m-2} - \;..$$

But by 1, (20), this expression is equal to zero; therefore, when a is a fraction, the product will be an expression composed of a finite number of terms, where the greatest exponent of a is $x+n$.

1. Actually performing the multiplication, the product will be found to be,

$$u_{x+1} a^{x+1} + \{u_{x+2} - nu_{x+1}\} a^{x+2} + \dots$$
$$+ \left\{ u_{x+n} - nu_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \dots nu_{x+1} \right\} a^{x+n}.$$

In future we shall put B′, B″,... B'^n, for the successive co-efficients of a in this expression.

(31.) Theorem.—$S_\infty^{x+1} u_x a^x = \frac{a^{x+1}}{(1-a)^n} S_{m=n}^{m=1} B'^m a^{m-1} = I_{x+1}$.

This result is obtained by dividing each side of the identity, (30), by $(1-a)^n$. (See Notation (1).)

1. Taking $x=0$ as the commencement of the series, then we have by subtraction,

$$S_x^1 u_x a^x = I_1 - I_{x+1},$$

which is the sum of the x first terms of the series.

2. By putting a^m for a, and multiplying by a^p, we readily obtain the expression for the more general form $S_x^1 u_x a^{mx+p}$.

The following theorem gives another method of summation.

(32.) Theorem.—$S_\infty^1 u_x a^{x+1} = A_0 \frac{1}{1-a} + A_1 \frac{1}{(1-a)^2} + \dots A_{n-1} \frac{[1]^{n-1}}{(1-a)^n}$, where the co-efficients are the same as in (19).

By expansion and multiplication,

$$\frac{[1]^{n-1}}{(1-a)^n} = [1]^{n-1} + [2]^{n-1} a + \dots [x]^{n-1} a^{x-1} + \dots$$

By 2, (19),

$$u_x a^{x-1} = a^{x-1} (A_0 + A_1 [x]^1 + \dots A_{n-1} [x]^{n-1})$$
$$\therefore S_\infty^1 u_x a^{x-1} = A_0 S_\infty^1 a^{x-1} + \dots A_{n-1} S_\infty^1 [x]^{n-1} a^{x-1}.$$

Summing by means of the above general form, the expression given in the theorem is obtained.

(33.) When the scale of relation of a recurring series is given, the sum may always be found by the following ***simple and direct*** method.

EXAMPLE.—Let $S=1+2x+8x^2+28x^3+\ldots$

where $2x^2$, $3x$ is the scale of relation.

Multiply each side by the sum of the terms of the scale of relation, viz., $2x^2+3x$; then

$$S(2x^2+3x)=3x+8x^2+28x^3+\ldots$$

where it will be observed, that after a certain number of terms, the latter series becomes identical with the given one, whence by subtraction,

$$S(1-2x^2-3x)=1-x; \therefore S=\frac{1-x}{1-2x^2-3x}$$

Illustration of Theorem (31).—To find the sum of the series

$$S^1_\infty x^2 a^x, \text{ or } a+4a^2+9a^3+\ldots$$

Here $u_x=x^2$, and $\therefore n=3$, $B'=(x+1)^2$, $B''=x+2)^2-3(x+1)^2$, and $B'''=(x+3)^2-3(x+2)^2+3(x+1)^2$, then taking $x=0$, the sum required $=\frac{a(1+a)}{(1-a)^3}$.

SECTION V.

ON THE SUMMATION AND TRANSFORMATION OF CERTAIN GENERAL TRIGONOMETRICAL AND OTHER FUNCTIONS.

(34.) THEOREM.—$S_{\infty}^{x+1}\, u_x\, a^x \cos x\theta =$

$$\frac{1}{(1-2a\cos\theta+a^2)^n}\, S_{m=n}^{m=1}\left\{a^{x+m}\,B'^m\, S_{p=n}^{p=0}\left(\frac{[-n]^p}{[1]^p}\, a^p \cos (x+m-p)\,\theta\right)\right\}$$

Because $\cos x\,\theta = \frac{1}{2}\left(z^x + \frac{1}{z^x}\right) \ldots (\alpha)$

$$\therefore\ S_{\infty}^{x+1}\, u_x\, a^x \cos x\,\theta = S_{\infty}^{x+1}\left\{\frac{1}{2}u_x\,(a\,z)^x + \frac{1}{2}\,u_x\left(\frac{a}{z}\right)^x\right\} =$$

$$\frac{\frac{1}{2}\,(a\,z)^{x+1}}{(1-a\,z)^n}\left(B'+B''\,(a\,z)'+\ldots B'^n\,(a\,z)^{n-1}\right)+$$

$$\frac{\frac{1}{2}\left(\frac{a}{z}\right)^{x+1}}{\left(1-\frac{a}{z}\right)^n}\left(B'+B''\left(\frac{a}{z}\right)'+\ldots B'^n\left(\frac{a}{z}\right)^{n-1}\right);$$ which result is obtained by summing the two series by (31).

Bring these two expressions on the right-hand member to the same denominator, that is, multiply the numerator of the first by the development of $\left(1-\frac{a}{z}\right)^n$, and the numerator of the second by the development of $(1-az)^n$, then, after addition, the *m*th, or general term of the numerator will become,

$$\frac{1}{2}\,B'^{m}\left\{(a\,z)^{x+m}+\left(\frac{a}{z}\right)^{x+m}-n\,a^{x+m+1}\left(z^{x+m-1}+\frac{1}{z^{x+m-1}}\right)+\right.$$

$$\left.\text{to } n \text{ terms.}\right\}=a^{x+m}\,B'^{m}\left\{\cos(x+m)\theta-n\,a\cos(x+m-1)\theta\right.$$

$$\left.+\frac{n\,(n-1)}{1.2}\,a^{2}\cos(x+m-2)\theta-\ldots\right\}$$

And the denominator will become, $(1-a\,z)^{n}\times\left(1-\frac{a}{z}\right)^{n}=$

$$\left(1-a\left(z+\frac{1}{z}\right)+a^{2}\right)^{n}=(1-2\,a\cos\theta+a^{2})^{n}.$$

From the general term just found the whole expression becomes known, whence, by abbreviation, the proposed theorem is established.

1. In like manner, the sum of $S_{\infty}^{x+1}\,u_{x}\,a^{x}\sin x\,\theta$, may be found from $\sin x\theta=\frac{1}{2\sqrt{-1}}\left(z^{x}-\frac{1}{z^{x}}\right),\ldots(\beta)$.

2. The series $S_{\infty}^{x+1}\,u_{x}\,a^{x}\cos^{n}x\theta$, may also be summed by this formula, since $\cos^{n}x\theta$ may be developed in terms of the cosine of multiple arcs. And so on to other trigonometrical functions.

3. Let $a=1$, then $1-2\,a\cos\theta+a^{2}=\left(2\sin\frac{\theta}{2}\right)^{2}$, and S_{∞}^{x+1}

$$u_{x}\cos x\theta=\frac{1}{\left(2\sin\frac{\theta}{2}\right)^{2n}}\,S_{m=n}^{m\;1}\left\{B'^{m}\,S_{p=n}^{p=0}\left(\frac{[-n]^{p}}{[1]^{p}}\cos\right.\right.$$

$$\left.\left.(x+m-p)\theta\right)\right\}$$

If K_{x+1} be put for this expression, then,

$S_{x}^{1}\,u_{x}\cos x\theta=K_{.}-K_{x+1}$.

4. Let $u_x=1$ in the general theorem, then $n=1$, $B'=1$, and B'', B''', ... $=0$, and $\therefore$

$$S_{\infty}^{x+1}\, a^x \cos x\theta = \frac{a^{x+1}\left(\cos (x+1)\,\theta - a\cos x\theta\right)}{1-2a\cos\theta+a^2}.$$

Putting Q_{x+1} for the sum just found,

$$S_x^{\,1}\, a^x \cos x\theta = Q_1 - Q_{x+1}.$$

Take $a=1$, and write $K\theta$ for θ, then, after a little reduction, we have,

$$S_x^{\,1} \cos x\,K\theta = \frac{\cos\dfrac{(x+1)\,K\theta}{2} \cdot \sin\dfrac{x\,K\theta}{2}}{\sin\dfrac{K\theta}{2}}$$

5. Let $\theta=0$, then the quantity expressed by the first symbol of summation will become $(1-a)^n$, and the whole may be readily reduced to the form of Theorem (31).

6. We shall give one example, in this place, of the general formula:—

Ex.—Required the sum of the infinite series $S_{\infty}^{x+1}\, xa^x \cos x\theta$.

Here, $n=2$, $B'=x+1$, $B''=x+2-2\,(x+1=-x$, $\therefore$ the sum required $=\dfrac{1}{(1-2a\cos\theta+a^2)^2}\left\{a^{x+1}\,(x+1)\left(\cos (x+1)\,\theta - 2a\cos x\theta + a^2\cos(x-1)\theta\right) - a^{x+2}\,x\left(\cos (x+2)\theta - 2\,a\cos (x+1)\theta + a^2\cos x\theta\right)\right\}$.

(35.) Theorem.—Let $fx=u_0+u_1\,x+u_2\,x^2+\ldots$; and put $f_n=fz^n x+fz^{-n}\,x$, then

$$u_0+u_1\cos^n\theta\,.\,x+u_2\cos^n 2\theta\,.\,x^2+\ldots=\frac{1}{2^n}\Big\{ f_n+nf_{n-2}+$$

$\frac{n(n-1)}{1.2} f_{n-4} + \ldots$ to $\frac{n+1}{2}$ terms $\Big\}$, when n is odd; and when n is even, the same expression is continued to $\frac{n}{2}$ terms $+ \frac{[n]^{\frac{n}{2}}_{-1}}{[1]^{\frac{n}{2}}} fx$; where it is to be observed that z is expressed by the same relation as in the last theorem.

In the equation for fx, first put zx for x, then $z^{-1}x$ for x in the same equation, and add; then, by observing the relation, α (34), we shall have $\frac{1}{2}\{fzx + fz^{-1}x\} = u_0 + u_1 \cos\theta \, . \, x + u_2 \cos 2\theta \, . \, x^2 + \ldots$

By performing this operation for n successive times, the theorem stated is obtained.

Similarly, if q_n be put for $f z^n x - f z^{-n} x$,

$u_1 \sin^n \theta \, . \, x + u_2 \sin^n 2\theta \, . \, x^2 + \ldots = \frac{1}{(2\sqrt{-1})^n} \Big\{ q_n - n\, q_{n-2} + \frac{n(n-1)}{1.2} q_{n-4} -$ to $\frac{n+1}{2}$ terms $\Big\}$; when n is odd; and when n is even this expression becomes,

$$\frac{1}{2^n(-1)^{\frac{n}{2}}} \Big\{ f_n - n f_{n-2} + \frac{n(n-1)}{1.2} f_{n-4} - \ldots + \frac{[-n]^{\frac{n}{2}}}{[1]^{\frac{n}{2}}} fx . \Big\}$$

1. To find the sum of the series $1 + \cos^n\theta . x + \ldots \cos^n\theta . x^{m-1}$.

Here $fx = \frac{1 - x^m}{1 - x}$, and $f_n = \frac{1 - z^{nm} x^m}{1 - z^n x} + \frac{1 - z^{-nm} x^m}{1 - z^n x} =$ $2\big(1 - x \cos n\theta - x^m \cos nm\theta + x^{m+1} \cos n(m-1)\theta\big) \div (1 - 2x \cos n\theta + x^2)$.

From which all the other fs are derived.

If m be taken infinite, i. e., if the series be continued to infinity, then $f_n = \frac{2(1-x\cos n\theta)}{1-2x\cos n\theta+x^2}$.

2. To find the sum of $1+\cos^n\theta \, . \, \frac{x}{1}+\cos^n 2\theta \, \frac{x^2}{1.2}+\ldots$

Here $fx=e^x$, and $f_n=e^{z^n x}+e^{z^{-n}x}=e^{x\cos n\theta}\{e^{x\sin n\theta\sqrt{-1}}+e^{-x\sin n\theta\sqrt{-1}}\}=2e^{x\cos n\theta}\cos(x\sin n\theta)$.

Let $n=1$, then

$$1+\cos\theta \, . \, \frac{x}{1}+\cos 2\theta \, \frac{x^2}{1.2}+\ldots=e^{x\cos\theta}\cos(x\sin\theta)$$

This result may be further verified by multiplying the developments of $e^{x\cos\theta}$ and $\cos(x\sin\theta)$.

Similar series may be derived from the second formula of the general theorem.

3. Let $n=1$, and $\theta=\frac{\pi}{2}$, then all the cosines of the odd arcs will be zero, while those of the even ones will be alternately minus and plus; hence,

$$\frac{1}{2}\{fzx+fz^{-1}x\}=u_0-u_2x^2+u_4x^4-\ldots$$

where $z^n+z^{-n}=2\cos\frac{n\pi}{2}$.

4. Let $fx=x+2x^2+3x^3+\ldots=\frac{x}{(1-x)^2}$; and $n=2$.

Here $\therefore \, S_{\infty}^{m=1}\, m\cos^2 m\theta \, . \, x^m=\frac{1}{2^2}\{f_2+2fx\}=$

$$\frac{1}{2}\left\{\frac{(x^3+x)\cos 2\theta-2x^2}{(x^2-2x\cos\theta+1)^2}+\frac{x}{(1-x)^2}\right\}$$

(36.) If we first multiply each side of the given function by x^m, and then proceed as already described, we shall have,

$$u_0 \sin^n m\theta + u_1 \sin^n (m+1)\theta . x + u_2 \sin^n (m+2)\theta . x^2 + \ldots =$$

$$\frac{1}{(2\sqrt{-1})^n}\left\{F_n - nF_{n-2} + \frac{n(n-1)}{1.2}F_{n-4} - \ldots \text{ to } \frac{n+1}{2} \text{ terms}\right\},$$

when n is odd; where $F_n = z^{nm} fxz^n - z^{-nm} fxz^{-n}$, and so on to the other cases.

Let $fx = \frac{1}{1-x}$, then

$$\sin m\,\theta + \sin (m+1)\theta . x + \ldots = \frac{\sin m\,\theta - x \sin (m-1)\,\theta}{1 - 2\,x \cos\theta + x^2}$$

If K_m be put for the left-hand member of this identity,

$$K_1 + K_2\,x + K_3\,x^2 + \ldots = \frac{\sin\theta\,(1-x^2)}{1-2x\cos\theta + x^2}.$$

(37.) Theorem. $u_0 \sin m\theta \sin v\theta + u_1 \sin (m+1)\theta \sin (v+1)\theta . x + \ldots = \frac{1}{4}\{2z^{v-m} fx - (z^{m+v} fx\,z^2 + z^{-(m+v)} fxz^{-2})\}$

and so on to other cases.

(38.) Theorem. $fx + f'x . \cos^n \theta . \frac{h}{1} + f''x . \cos^n 2\theta . \frac{h^2}{1.2} + \ldots =$

$$\frac{1}{2^n}\left\{f(x + z^n h) + nf(x + z^{n-2} h) + \frac{n(n-1)}{1.2} f(x + z^{n-4} h) + \ldots\right\}$$

By Taylor's Theorem,

$$f(x+h) = fx + f'x . h + f''x . \frac{h^2}{1.2} + \ldots$$

In this identity, put first zh for h, and afterwards $z^{-1} h$ for h, then by adding the resulting equations we have,

$\frac{1}{2}\{f(x+zh)+f(x+z^{-1}h)\}=fx+f'x.\cos\theta.h+f''x.\cos 2\theta.$

$\frac{h^2}{1.2}+\ldots$ By repeating the operation for n times the theorem will be obtained.

1. Let $f(x+h)=\log.(1+h)=h-\frac{h^2}{2}+\frac{h^3}{3}-\ldots$, where the derived functions are obvious. Making the proper substitutions, the left-hand member of the theorem becomes,

$$\frac{1}{2^n}\left\{\log.(1+z^n h)+n\log.(1+z^{n-2}h)+\frac{n(n-1)}{1.2}\log.(1+z^{n-4}h)+\ldots\right\}=$$

$$\frac{1}{2^n}\log.\left\{(h^2+2h\cos n\theta+1)\,(h^2+2h\cos(n-2)\theta+1)^n\,(h^2+2h\cos(n-4)\theta+1)^{\frac{n(n-1)}{2}}\ldots.\right\}$$

where the factors must be continued to $\frac{n+1}{2}$ terms when n is odd; and to $\frac{n}{2}$ factors with the addition of

$\frac{[n]^{\frac{n}{2}|-1}}{[1]^{\frac{n}{2}}}\log.(1+h)$, when n is even. This expression, therefore, is the sum of the infinite series,

$$\cos^n\theta.h-\cos^n 2\theta.\frac{h^2}{2}+\cos^n 3\theta.\frac{h^3}{3}-\ldots$$

This result may be readily verified by making $\theta=2\pi$, for then this series becomes the development of log. $(1+h$; but

in this case the expression for the sum, when n is odd, may be reduced to,

$$\frac{1}{2^n}\left(1+n+\frac{n(n-1)}{1.2}+\ldots\right)\log.(1+h)=\log.(1+h).$$

Ex. Let $n=1$, then,

$$\frac{1}{2}\log.(h^2+2h\cos\theta+1)=\cos\theta\,.\,h-\cos 2\theta\,.\,\frac{h^2}{2}+\cos 3\theta\,.$$
$$\frac{h^3}{3}-\ldots$$

Similarly the sum of a series, containing the sines of multiple arcs, may be obtained.

(39.) Theorem.—

$$S_{n=n}^{n=1} fn\,\theta = S_{\infty}^{k=1}\left\{u_k x^k \frac{\cos\frac{(n+1)k\theta}{2}\,.\,\sin\frac{n\,k\theta}{2}}{\sin\frac{k\theta}{2}}\right\};$$

where the form of the function is derived from the sum of the series,

$$f\theta = S_{\infty}^{k=1}(u_k x^k \cos k\theta)$$

$\therefore$ putting $n\theta$ for θ,

$$fn\,\theta = S_{\infty}^{k=1}(u_k x^k \cos kn\theta)$$

Taking n successively $=1, 2, 3, \ldots n$, we have by addition, or what is the same thing, performing the operation of summation between the limits $n=1$, and $n=n$,

$$S_{n=n}^{n=1} fn\,\theta = S_{n=n}^{n=1}\{S_{\infty}^{k=1}(u_k x^k \cos kn\,\theta)\}$$
$$= S_{\infty}^{k=1}\{S_{n=n}^{n=1}(u_k x^k \cos kn\,\theta)\}$$

where the second step is obtained by inverting the order of

summation. The theorem is derived from the last expression by performing one of the summations. See (34), 4.

1. Similarly it may be shown, from Taylor's Theorem, that

$$S_{n=n}^{n=0} f(x+n) = S_{k=m}^{k=0} \left\{ \frac{f'^k x}{[1]^k} S_{n=n}^{n=0} n^k \right\}$$

where $f'^k x$ is the kth derived function, and m the degree of the given function.

This theorem will always enable us to express the sum of a function in terms of the sum of the simple powers of a variable.

The principle of inverting the order of summation, is of universal application, and is rendered sufficiently obvious by the preceding demonstration.

(40.) THEOREM.

$$f\frac{2\pi}{n} + f\frac{4\pi}{n} + \ldots f\frac{2n\pi}{n} = n\{u_n x^n + u_{2n} x^{2n} + u_{3n} x^{3n} + \ldots\}$$

In Theorem (39), take $\theta = \frac{2\pi}{n}$; then the trigonometrical expression in the second member will be zero for all values of k not divisible by n, and, in all other cases, the expression will become n; hence the theorem is obvious.

Ex.—Let $f\theta = e^{x \cos\theta} \cos(x \sin\theta) - 1$; See Ex. 2, (35), and $n=4$, then $u_n = \frac{1}{[1]^4}$, and by a little reduction, we find,

$$\frac{1}{4}(e^x + e^{-x} + 2\cos x) = 1 + \frac{x^4}{[1]^4} + \frac{x^8}{[1]^8} + \ldots$$

(41.) THEOREM. $S^1{}_m \frac{fx \, . \, t^x}{[a+rx]_r^{n+1}} = \frac{1}{[1]^n r^n} \Big\{ z \Big(k_0 - \frac{n_1 k_1}{t}$

$$+\frac{n_2 k_2}{t^2}-\ldots\Big)+n_1 k_1(Q_1-q_1)-n_2 k_2(Q_2-q_2)+\ldots\Big\}$$

Where $z=S^1{}_m\dfrac{t^x}{a+rx}$; $Q_p=\dfrac{1}{a+pr}+\dfrac{1}{t(a+r(p-1))}+$

$\ldots\dfrac{1}{t^{p-1}(a+r)}$; $q=t^{m-p+1}\Big(\dfrac{1}{a+r(m+1)}+\dfrac{t}{a+r(m+2)}+$

$\ldots\dfrac{t^{p-1}}{a+r(m+p)}\Big)$; and k_m has the same meaning as in (14).

Multiply the second form given in (14), by t^x, and then take x successively 1, 2, 3, ... m, and add the corresponding columns, or, what is the same thing, take the sum of the general term, between the limits 1 and m; then the proposed series will be expressed by

$$\frac{1}{[1]^{n|r^n}}\Big\{k_0\, S^1{}_m\frac{t^x}{a+rx}-n_1\, k_1\; S^1{}_m\,\frac{t^x}{a+r(x+1)}+\ldots\Big\}$$

We proceed now to express this sum in terms of z; and in order to effect this, it will be necessary to observe, that the series, or columns, towards the right of the first, have certain terms in deficiency at the commencement of the series, and a corresponding number of terms in excess at the end, hence the pth column will be found to be,

$$(-1)^p\Big\{z\frac{n_p k_p}{t^p}-n_p\, k_p\,(Q_p-q_p)\Big\};$$

whence the theorem is obvious.

If m be taken infinite the qs vanish, and in this case the terms of the proposed series are continued ad infinitum.

Cor. 1.—In order to eliminate z, put its co-efficient$=0$, which will give us an equation of condition, from which we may determine one of the constants in fx.

For this purpose we have,

$$k_m = f\left(-\frac{a+mr}{r}\right) = a_0 - a_1\left(\frac{a+mr}{r}\right) + \dots \pm a_n\left(\frac{a+mr}{r}\right)^n =$$

$a_0 - v_m$, for the sake of abbreviation.

Substitute these values of the *k*s, in the equation of condition, and then proceed to determine a_0, which after a little reduction will become,

$$a_0 = \left(v_0 - \frac{n_1 v_1}{t} + \frac{n_2 v_2}{t^2} - \dots\right) \div \left(1 - \frac{1}{t}\right)^n.$$

In this case, therefore,

$$S^1{}_m \frac{fx \,.\, t^x}{[a+rx]_r^{n+1}} = \frac{1}{[1]^{n_r n}} \{n_1 (a_0 - v_1)(Q_1 - q_1) - n_2(a_0 \quad v_2)$$

$$(Q_2 - q_2) + \dots\}$$

Ex. $S^1{}_m \dfrac{x+a_0}{[1+2x]_2{}^2} \cdot \dfrac{1}{3^x}$.

Here $n_1 = 1$; $t = \frac{1}{3}$; $k_m = a_0 - \dfrac{1+2m}{2}$; and $\therefore v_m = \dfrac{1+2m}{2}$;

$$a_0 = \left(\frac{1}{2} - \frac{3}{2} \cdot 3\right) \div -2 = 2$$

Whence the sum required $= \dfrac{1}{2}\left\{\dfrac{1}{2}\left(\dfrac{1}{3} - \dfrac{1}{1+2(m+1)} \cdot \dfrac{1}{3^m}\right)\right\} =$

$$\frac{1}{12} - \frac{1}{4+8(m+1)} \cdot \frac{1}{3^m}.$$

Cor. 2. If $t=1$, and the function in the numerator be of the $(n-1)$th degree, then, by 1, (20), the co-efficient of z vanishes, and the remaining expression, under these restrictions, is the sum of the series given in (25.)

Cor. 3. Take $a=0$, $r=1$, and $m=\infty$, then $q_p=0$, $z=\log.\frac{1}{1-t}$, and, substituting these particular values, the right-hand member of the theorem gives the sum of the series $S^1_\infty \frac{fx \,.\, t^x}{[x]^{n+1}}$.

Cor. 4. If $fx=1$ in the last result, then it will be found that,

$$S^1_\infty \frac{t^x}{[x]^{n+1}} = \frac{1}{[1]^n}\left\{z\left(1-\frac{1}{t}\right)^n + n_1 Q_1 - n_2 Q_2 + n_3 Q_3 - \ldots\right\}$$

where in this case $Q_p = \frac{1}{t^0 p} + \frac{1}{t(p-1)} + \ldots \frac{1}{t^{p-1}}$

Ex. $\frac{1}{1.2.3}\cdot\frac{1}{2} + \frac{1}{2.3.4}\cdot\frac{1}{4} + \ldots = \frac{1}{2}\log.2 - \frac{1}{4}$

Here $n_1=2$, $n_2=1$, and $t=\frac{1}{2}$.

(42). THEOREM.—

$$S^1_\infty \frac{v_x}{[(x)^2]^{n+1}} = s\,(A_0 + \ldots A_n) - (A_1 s_1 + \ldots A_n s_n) +$$

$$\frac{1}{[1]^n}\left\{u_1 + \frac{1}{2}(u_2 - n_1 u_1) + \ldots \frac{1}{n}(u_n - n_1 u_{n-1} + n_2 u_{n-2} - \ldots)\right\}$$

where v_x does not exceed the $2n$th degree; $s = \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots$; $s_m = \frac{1}{1^2} + \ldots \frac{1}{m^2}$; $A_m = n_m{}^2\, v_{-m} \div [(1)^2]^n$;

$$\text{and } u_m = \frac{v_m}{[m]^{n+1}} - [m]^{n+1}\left(\frac{A_0}{m^2} + \ldots \frac{A_n}{(m+n)^2}\right).$$

Assume,

$$\frac{v_x}{[(x)^2]^{n+1}} = \frac{A_0}{x^2} + \ldots + \frac{A_n}{(x+n)^2} + \frac{u_x}{[x]^{n+1}};$$

where u_x must be a function of the $(n-1)$th degree.

The co-efficients A_0, A_1, ... A_n are determined by the method used in (19); then, by transposition, &c., u_x becomes a known function of x, which, owing to the manner in which the co-efficients, A_0, A_1, ..., have been derived, will also be integral. But, however, for our present purpose it is not necessary to find the function in this form.

Take now the sum of this general term, between the limits $x=1$, and $x=\infty$; then, by arranging the terms, as in (42), and summing the last term by (27), the expression given in the theorem will be obtained.

Similarly the sum is found, when the factorial increment is r. See (3).

Example.—To find the sum of the series $S^1_\infty \frac{x^4}{[(x)^2]^3}$.

Here $v_{-m}=m^4$, $n=2$, $A_0=0$, $A_1=1$, $A_2=4$, $u_2=\frac{2^4}{[2]^3}$

$-[2]^3\left(\frac{1}{3^2}+\frac{4}{4^2}\right)=-8$, $u_1=-4$,

and the sum of the series$=5s-8$.

Cor. 1. In order that the sum of the series may be expressed in finite terms, put the co-efficient of s equal to nothing, and let $v_x=K_x+e$, where $K_0=0$; then, after reduction, we have,

$$e=-\frac{n^2{}_1 K_{-1}+n^2{}_2 K_{-2}+\ldots}{1+n^2{}_1+n^2{}_2+\ldots};$$

which is the equation of condition; and with this restriction

the remaining part of the expression, in the theorem, will give the sum of the series.

Obs.—A distinguished analyst, Mr. Woolhouse, gave the sum of a particular case of this series, in the Ladies' Diary for the year 1836; and Mr. Rutherford discussed the same form in the subsequent number, with his usual elegance and success; at the same time it may not be improper to observe, that the latter mathematician has committed an error, arising no doubt from inadvertency, in supposing that he has given "a general expression for the sum of the squares of the reciprocals of any order of figurate numbers."

(43.) THEOREM.—

$$f\ x+k)=A_0 fx+\ldots+A_m f'^m(x-mpk)+\ldots;$$

where $A_m=\dfrac{k^m\ (1+mp)^{m-1}}{[1]^m}$, and p is an arbitrary quantity not appearing in the given function.

Let $fx=a_0+\ldots+a_n x^n$; then, by Taylor's Theorem,

$$f(x+k)=f(k)+\ldots+f'^{\,n-m}(k)\,.\,\frac{x^{n-m}}{[1]^{n-m}}+\ldots\ (\alpha)$$

Let the proposed expansion be assumed, then we shall have to determine the co-efficients A_0, A, &c., so as to render the expression identical with (α); for this purpose, expand all the terms by the form,

$$f'^m(x-mpk)=f'^m(-mpk)+\ldots+f'^n(-mpk)\ .\ \frac{x^{n-m}}{[1]^{n-m}};$$

then the co-efficient of $\dfrac{x^{n-m}}{[1]^{n-m}}$, will be,

$$A_m\ f'^n(-mpk)+A_{m-1}\ f'^{n-1}(-\overline{m-1}\,pk)+\ldots$$
$$+A_0 f'^{n-m}(0)=f'^{n-m}(k).$$

In this identity take m successively 0, 1, 2, ..., and calculate the derived functions from the development of fx, then,

$$A_0 f'^n(0) = f'^n(k)\ ; \text{ or } A_0 = 1,$$

$$A_1 = \frac{f'^{n-1}(k) - f'^{n-1}(0)}{f'^n(-pk)} = \frac{[2]^{n-1} a_n k}{[1]^n a_n} = k,$$

and so on to the other co-efficients.

Cor. 1. When p is taken unity, we obtain a theorem, which, I believe, was first discovered by Mr. Murphy.

Cor. 2. If $fx = x^n$, we then have,

$$(x+k)^n = x^n + \ldots + n_m k^m (1+mp)^{m-1} (x-mpk)^{n-m} + \ldots$$

In the place of deriving this result, as a corollary from the general theorem, the student is recommended to establish it from first principles, which may readily be done by operating with the binomial development, as we have done with Taylor's.

Cor. 3. If $x=1$, and $k=-1$, this last identity becomes a particular form of 1, (20).

Cor. 4. Put $-k$ for k, in the general theorem, then the signs of the terms, in the left-hand member, will be alternately plus and minus. By making $p=0$, this expression becomes Taylor's Theorem.

Various other neat particular forms may be obtained by,

1. Taking $p=k=x$; 2. Taking $p=\frac{1}{k}$; 3. Putting $x=0$ in Cor. 3; 4. Making $x=1$, and $k=-1$.

(44.) Theorem.

$$\mathrm{F}(0) f(x) - \ldots (-1)^m \mathrm{F}(ma) f'^m (x+ma) \cdot \frac{x^m}{[1]^m} \pm \ldots =$$

$$F(0)f(0)-\ldots(-1)^m\Delta^m\{F(0)f'^m(0)\}\cdot\frac{x^m}{[1]^m}\pm\cdots$$

By development, the left-hand member becomes

$$S_{\infty}^{m=0}(-1)^m F(ma)\cdot\frac{x^m}{[1]^m}\Big\{f'^m(ma)+\ldots+f'^{m+n}(ma)\cdot$$
$$\frac{x^n}{[1]^n}+\cdots\Big\}$$

Arranging the terms of this expression according to the powers of x, we readily find the mth term in the following form,

$$\frac{x^m}{[1]^m}\{F(0)f'^m(0)-\ldots\pm m_n F(na)f'^m(na)\mp\ldots\}$$

which, expressed by the symbol of differences, becomes the general term of the left-hand member of the theorem.

Cor. 1. This theorem assumes a neat form, by making $Fx=1$, and $a=1$.

By precisely the same process of reasoning, the succeeding theorem may be established.

(45.) Theorem.

$$u_0 f(a-x)+\ldots+u_m f'^m(a-x)\cdot\frac{x^m}{[1]^m}+\ldots=$$

$$u_0 fa+\ldots+\Delta^m u_0\cdot f'^m a\cdot\frac{x^m}{[1]^m}+\ldots \quad \text{Or, } f(a+x\,\Delta)\,u_0.$$

Cor. 1. Putting $z+x$ for a, we find,

$$u_0 fz+\ldots+u_m f'^m z\cdot\frac{x^m}{[1]^m}+\ldots=f(\overline{z+x}+x\,\Delta)u_0.$$

Cor. 2. If $u_m=m^n$, in the last identity, we have,

$$0^n fz+1^n f'z\cdot x+2^n f''z\cdot\frac{x^2}{1.2}+\ldots=f(z+x+x\,\Delta)0^n$$

When $z=0$, and $x=1$, this expression becomes,

$$f'\,0\,.\,1^n+f''\,0\,.\,\frac{2^n}{1.2}+\ldots=f\,(1+\Delta)\,0^n$$

Let $fx=e^x$, then, changing the symbols of operation to those of quantity,

$$\frac{1^n}{1}+\frac{2^n}{1.2}+\frac{3^n}{1.2.3}+\ldots=e\left\{\frac{\Delta\,0^n}{1}+\frac{\Delta^2\,0^n}{1.2}+\ldots\right\}$$

Ex. If $n=3$,

$$\frac{1^3}{1}+\frac{2^3}{1.2}+\frac{3^3}{1.2.3}+\ldots=5e$$

Cor. 3. In the last general expression take $z=0$, and let $fz=\frac{1}{1-x}$, then, after a little reduction, we find,

$$1^n\,.\,x+2^n\,.\,x^2+3^n\,.\,x^3+\ldots=\frac{1}{1-x}\left\{\Delta\,0^n.\frac{x}{1-x}+\Delta^2\,0^n\left(\frac{x}{1-x}\right)^2+\ldots\right\}$$

Cor. 4. Let s_n be put for the sum of the last series, then by proceeding as in 1, (40), we have,

$$S^{n=1}f\,(x+n)\,a^n=S^{n=0}_{=m}\frac{f'^n x}{[1]^n}\,.\,s_n$$

If $x=0$, then,

$$f1\,.\,a+f2\,.\,a^2+f3.a^3+\ldots=s_0f0+\ldots+s_n\frac{f'^n\,0}{[1]^n}+\ldots$$

Cor. 5. Multiplying the identity in Cor. 3, by the development of $\frac{1}{1-x}$, we have,

$$S^{m=1}_{\infty}\,x^m\,S^{m=1}_{=m}\,m^n=\frac{s_n}{1-x}$$

Cor. 6. By taking $a=x$, and $fx=\frac{1}{1-x}$, we readily obtain Euler's Theorem for the transformation of series.

(46.) THEOREM.

$$f(z+xa^z)=fz+A_1 x+A_2 x^2+\ldots,$$

where $A_n=\frac{1}{[1]^n} S^{m=0} n_m (m A)^{n-m} f'^m z$, and n_m has the same meaning as in (27.)

By Taylor's Theorem the proposed function becomes,

$$fz+\ldots+f'^n z.\frac{x^n a^{nz}}{[1]^n}+\ldots$$

Let the exponentials be developed, and, after multiplication, let the co-efficients of x^n be collected, then the common factors being taken out so as to produce the binomial co-efficients, we shall find the nth co-efficient as stated in the theorem.

Cor. 1. Let $a=e, fz=e^z$, and $z=0$, then

$$e^{xe^x}=1+A_1 x+A_2 x^2+\ldots; \text{ where}$$

$$A_n=\frac{1}{[1]^n}\{0^n+n_1 1^{n-1}+n_2 2^{n-2}+\ldots\}.$$

Comparing this co-efficient with the corresponding co-efficient in Herschel's theorem, we have,

$$0^n+n_1 1^{n-1}+n_2 2^{n-2}+\ldots=(1+\Delta)^{1+\Delta} 0^n$$

(47.) THEOREM.—

$$f(z+va^z)=f(z+v)+\ldots+A_n z^n+\ldots.$$

$$\text{where, } A_n=\frac{A^n}{[1]^n}f\,(\overline{z+v}+v\Delta)0^n.$$

Proceeding as in the last theorem, we find,

$$\mathrm{A}_n = \frac{\mathrm{A}^n}{[1]^n}\left\{1^n f'z \,.\, v + \ldots + m^n\, f'^m z \,.\, \frac{v^m}{[1]^m} + \ldots\right\}$$

But by Cor. 2, (45), this expression is equal to the formula stated in the theorem.

Cor. 1. If $z=0$, $v=1$, and $a=e$, we obtain Sir John Herschel's celebrated theorem.

(48.) Theorem. $f\{e^x(1+\Delta)\}\,0^n = f(1+\Delta)\,0^n \,.\, e^{0\,z}$

In Theorem (47), take $z=0$, $a=e$, and $v=e^x$, then the left-hand member of the proposed theorem will be the co-efficient of $\frac{x^n}{[1]^n}$ in the development of fe^{x+z}.

In the development of fe^x, Cor. 1, let x become $x+z$, then by expanding the right-hand member in the ascending powers of x, the co-efficient of $\frac{x^n}{[1]^n}$ will be,

$$f(1+\Delta)\,0^n + \ldots f(1+\Delta)\,0^{n+m} \,.\, \frac{z^m}{[1]^m} + \ldots =$$

the right-hand member of the theorem. Therefore, by equating, the proposed identity is established.

(49.) Theorem.—

$$\Delta^m f^n a + \Delta^m f^{n+1} a \,.\, \frac{\Delta^n x^{n+1}}{[1]^{n+1}} + \Delta^m f^{n+2} a \,.\, \frac{\Delta^n x^{n+2}}{[1]^{n+2}} + \ldots =$$

$$\Delta^n f^m a + \Delta^n f^{m+1} a \,.\, \frac{\Delta^m x^{m+1}}{[1]^{m+1}} + \Delta^n f^{m+2} a \,.\, \frac{\Delta^m x^{m+2}}{[1]^{m+2}} + \ldots$$

Because either side of this identity is equal to $\Delta^{n+m} f(a+x)$; observing that all the differences of the powers of x below the nth vanish, whilst the nth difference of the nth power is $[1]^n$.

Cor. 1. The theorem assumes a neat form by making $x=0$.

Cor. 2. When $m=0$, and $x=0$, the theorem becomes the same as that given in Hymer's Diff. Eq., p. 21.

(50.) THEOREM. If f, f_1, f_2, &c., be put for functions of x, then,

$$(f.f_1.f_2\ldots)'^n=(f+f_1+f_2+\ldots)^n;$$

where n, in the left-hand member, is a symbol of operation, giving, in the development, the orders of the derived functions; observing also, that as Newton's development may be written, $(a+b+\ldots)^n=a^n.b^0.c^0\ldots+$ &c., so the same thing is to be attended to in reference to the proposed theorem.

First take $Fx=fx.f_1 x$, and in this equality let x become $x+h$, then, expanding by Taylor's theorem,

$$F(x+h)=Fx+\ldots+F'^n x.\frac{h^n}{[1]^n}+\ldots$$

Expanding the two functions on the right-hand in the same manner, and multiplying, we readily find the co-efficient of h^n, hence, by equating,

$$F'^n x=f_1 f'^n+n_1 f'_1 f'^{n-1}+n_2 f'^2_1 f'^{n-2}+\ldots\ (1)$$
$$=(f+f_1)^n,\text{ that is,}$$

$$(f.f_1)'^n=(f+f_1)^n\ldots\ (2)$$

For f_1, put $f_1.f_2$, in (1), and in the resulting expression substitute the values contained in Eq. (2), then,

$$(f.f_1.f_2)'^n=f_1 f_2 f'^n+n_1(f_1+f_2)'f'^{n-1}+\text{\&c.}$$
$$=(f+f_1+f_2)^n;$$

and so on to the full extent of the enunciation.

(51.) **Theorem.**

$$f0+(m+1)_1 f'\,0\,.\,x+(m+2)_2 f''\,0\,.\,\frac{x^2}{1.2}+\ldots=$$

$$fx+m_1 f'x\,.\,x+m_2 f''x\,.\,\frac{x^2}{1.2}+\ldots;$$

where $(m+n)_p$ is the pth binomial co-efficient, the exponent being $m+n$.

By Maclaurin's theorem, multiplying by x^m, and then putting $x+h$ for x, we have,

$$(x+h)^m f(x+h)=f0\,(x+h)^m+f'\,0\,(x+h)^{m+1}+\ldots$$

The theorem will be obtained from this expression by expanding, and then equating the co-efficients of h^m.

Cor. 1. If $Fx=x^m fx$, then by Eq. (1), (51), the right-hand member $=\dfrac{F'^m x}{[1]^m}$

Cor. 2. Let $fx=\sin x$, then $f0=0$, $f'0=1$, &c., and,

$(m+1)_1\,x-(m+3)_3\,\dfrac{x^3}{[1]^3}+$ &c., $=\sin x+m_1\cos x\,.\,x-$

$m_2\sin x\,.\,\dfrac{x^2}{1.2}+\ldots$

Ex. If $m=2$, we have,

$3\,.\,2\,.\,x-5\,.\,4\,.\,\dfrac{x^3}{[1]^3}+7\,.\,6\,.\,\dfrac{x^5}{[1]^5}-\ldots=2\Big(\sin x+2\cos x\,.\,x-$

$\sin x\,.\,\dfrac{x^2}{2}\Big)$.

(52.) Let $D\,\{fx\}$ point out the operation performed upon the function in order to obtain the expression in the last theorem, then,

$$f0+(m+1)^n{}_1 f'0\,.\,x+(m+2)^n{}_2 f''0\,.\,\frac{x^2}{1.2}+\ldots=D^n\,\{fx\}$$

(53.) **Theorem.** If $y=fx$ be the equation of a curve, $c=$the distance between two ordinates, and a the distance of one of them from the origin of co-ordinates; then, when n is infinite, the area lying between the two ordinates $=$

$$\frac{c}{n}\,S^{m=1}_{=n} f\left(a+\frac{mc}{n}\right)$$

This result may be readily established by dividing the curve surface into n parts, by drawing $n-1$ equi-distant ordinates, and then observing that each part approaches to a rectangle as n is increased.

Ex. Let $y=\frac{x^2}{p}$ be the equation to the curve, and $a=0$.

Here, the area $=\frac{c}{n}\,S^{m=1}_{=n}\left(\frac{mc}{n}\right)^2.\frac{1}{p}$

Taking the sum of the series (Ex. 2, p. 24), this expression becomes $\frac{c^3}{n^3.p}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)=\frac{c^3}{3p}$.

54.) **Theorem.**

$$f\{\log.(1+\Delta)\}^m 0^{2nm}=\frac{1}{2}\left(f\{\log.(1+\Delta)+f\left\{\log.\frac{1}{1+\Delta}\right\}\right)0^{2n}$$
$$\times\ [2\,n+1]^{2n(m-1)}.$$

The co-efficient of t^{2n} in the development of $\frac{1}{2}\,\{f(t)+f(-t)\}$, will be the same as the co-efficient of t^{2nm} in that of $f\,(t^m)$, therefore, by Herschel's Theorem, and equating, the proposed identity is established.

(55.) **Theorem.** $f\,\{\log.\,(1+\Delta)\}^m\,0^n=0$,

where n is not a multiple of m, because, in the development of $f\,(t^m)$, all the co-efficient swill be zero, excepting the 1st, mth, $2m$th, &c.

(56.) THEOREM.

$$f'^{n} x = f\left(x + \{\log.\,(1+\Delta)\}^{m}\right) 0^{nm} \div [n+1]^{n(m-1)}$$

This expression for the nth derived function, is obtained by equating the co-efficient of t^{nm}, in the development of $f\,(x+t^{m})$, with the corresponding co-efficient obtained by Herschel's Theorem.

Cor. 1. When $m=1$, we have,

$$f'^{n} x = f\{x + \log.\,(1+\Delta)\}\, 0^{n}$$

Cor. 2. In the last result, take $fx = x^{m}$, and $x=1$, then,

$$[m-n+1]^{n} = \{1 + \log.\,(1+\Delta)\}^{m}\, 0^{n}$$

When $n = \frac{1}{2}$, and $m=0$, the left-hand member becomes $\frac{1}{\sqrt{\pi}}$; see 4, (10). This circular function, therefore, in the absence of higher evidence, may be *presumed* to be the interpretation of $\{1 + \log.\,(1+\Delta)\}^{0}\, 0^{\frac{1}{2}}$.

THE END.

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